International Journal of Modern Physics B
Vol. 29, No. 19 (2015) 1550124 (27 pages)

DOI: 10.1142/S0217979215501246

A recursive approach for geometric quantifiers of quantum correlations in multiqubit Schrödinger cat states

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Received 16 January 2015
Accepted 19 April 2015
Published 19 June 2015

A recursive approach to determine the Hilbert–Schmidt measure of pairwise quantum discord in a special class of symmetric states of k qubits is presented. We especially focus on the reduced states of k qubits obtained from a balanced superposition of symmetric n-qubit states (multiqubit Schrödinger cat states) by tracing out $n-k$ particles ($k = 2, 3, \ldots, n-1$). Two pairing schemes are considered. In the first one, the geometric discord measuring the correlation between one qubit and the parity grouping $(k-1)$ qubits is explicitly derived. This uses recursive relations between the Fano–Bloch correlation matrices associated with subsystems comprising $k, k-1, \ldots$ and two particles. A detailed analysis is given for two-, three- and four-qubit systems. In the second scheme, the subsystem comprising the $(k-1)$ qubits is mapped into a system of two logical qubits. We show that these two bipartition schemes are equivalents in evaluating the pairwise correlation in multiqubits systems. The explicit expressions of classical states presenting zero discord are derived.

Keywords: Coherent states; Dicke states; GHZ and W states; geometric quantum discord; Hilbert–Schmidt distance.

PACS numbers: 03.65.-w, 03.67.-a, 03.65.Aa, 03.67.Mn

1. Introduction

Quantum correlations in multipartite systems have generated a lot of interest during the last two decades.[1–3] This is essentially motivated by their promising

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applications in the field of quantum information such as implementing quantum cryptographic protocols, speeding up quantum computing algorithms and many more quantum tasks (see for instance Refs. 4 and 5). An important issue in investigating quantum correlations concerns the appropriate measure to decide about quantumness in a given quantum system and to separate between classical and quantum states. The characterization of quantum correlations is necessary in order to exploit their advantages, in an efficient way, in the context of quantum information processing such as quantum teleportation,[6] superdense coding[7] and quantum key distribution.[8] Several methods and different measures of quantum correlation were exhaustively discussed in the literature from various perspectives and for many purposes (for a recent review see Ref. 3). They can be classified in two main categories: Entropic-based measures and geometric quantifiers or norm-based measures. Entanglement of formation, linear entropy, relative entropy and quantum discord[9–14] constitute familiar entropic quantifiers of correlations. Probably, quantum discord, which goes beyond entanglement, is the most prominent of these correlations. It has been the subject of intensive studies during the last decade. It was originally defined as the difference between two quantum analogs of the classical mutual information.[13,14] The explicit evaluation of based entropy measures require optimization procedures which are in general very complicated to achieve. This constitutes the main obstacle in order to get computable expressions of quantum correlations. To overcome such difficulties, geometric measures, especially the ones based on Hilbert–Schmidt norm, were considered to formulate a geometric variant of quantum discord.[15] The Hilbert–Schmidt distance was used to quantify classical correlations.[16,17] We notice that the measure of quantum and classical correlations in bipartite systems can be also evaluated through the 1-norm distance (trace distance).[18–21]

On the other hand, the extension of Hilbert–Schmidt measure of quantum discord to d-dimensional quantum systems (qubits) was reported in Refs. 22–24 (see also Ref. 25 and references quoted therein). It must be emphasized that this higher-dimensional extension can be adapted to understand the pairwise quantum correlations in multiqubit systems. Indeed, geometric quantum discord based on the Hilbert–Schmidt norm turns out to be more tractable, in multiqubit systems, from a computational point of view than entropic-based measures. In this sense, we employ the approach by Dakic *et al.*[15] to investigate the quantum correlations in mixed multiqubit states. Specifically, we shall consider a balanced superposition of symmetric multiqubit states in which the symmetry properties offer drastic simplification in evaluating quantum correlations.

This paper is organized as follows. In Sec. 2, we discuss the relevance of symmetric multiqubit (n-qubit) states in defining Schrödinger cat states. We shall essentially focus on balanced superpositions, symmetric or antisymmetric under the parity transformation, which coincide with even and odd spin atomic coherent states. A special attention, in Sec. 3, is devoted to reduced states describing subsystems containing k qubits ($k = 2, 3, \ldots, n-1$) obtained by tracing out $n-k$

qubits from a n-qubit Schrödinger cat state. This trace procedure gives rise to states called extended X states. The algebraic structure of such states provides a nice prescription to evaluate the quantum correlation based on Hilbert–Schmidt (geometric quantum discord) between one qubit and $(k-1)$ qubits contained in a mixed state. This procedure is explicitly described in Sec. 4. We consider in detail the cases of two- and three-qubit systems. We develop the general method to determine analytically geometric discord in mixed k-qubit states. We also derive the explicit forms of classical (zero discord) states. In Sec. 5, we introduce another scheme according to which the second part of the system containing $k-1$ qubits is mapped into two logical qubits. In this picture, the whole system reduces to a two-qubit system. Remarkably, the geometric measure of quantum discord obtained, in this second scheme, coincides with one derived in the first bipartition scheme (Sec. 4). As illustration, a detailed analysis is given for $k=3$ and $k=4$. The method developed in this paper which extends the geometric measure of two-qubit X states to embrace k-qubit X states is useful in investigating the global pairwise correlation in multipartite qubit systems. Concluding remarks close this paper.

2. Symmetric Multiqubit Systems

The multiqubit symmetric states were shown relevant for different purposes in quantum information science.[26–33] In this paper, we shall mainly focus on an ensemble of n-spin-1/2 systems prepared in even and odd spin coherent states.

2.1. *Spin coherent as symmetric multiqubit systems*

We consider n identical qubits. Each qubit lives in a two-dimensional Hilbert space $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$. The Hilbert space of the n-qubit system is given by n tensored copies of $\mathcal{H}$:

$$\mathcal{H}_n := \mathcal{H}^{\otimes n}\,.$$

Among the multipartite states in $\mathcal{H}_n$, multiqubit states obeying exchange symmetry are of special interest from experimental as well as mathematical point of views. An arbitrary symmetric n-qubit state is commonly represented in either Majorana[34] or Dicke[35] representation. Any multiqubit state, invariant under the exchange symmetry, is specified in the Majorana description by the state (up to a normalization factor):

$$|\psi_s\rangle = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} |\eta_{\sigma(1)},\ldots,\eta_{\sigma(n)}\rangle\,, \tag{1}$$

where each single qubit state is $|\eta_i\rangle \equiv (1+\eta_i\bar{\eta}_i)^{-1/2}(|0\rangle + \eta_i|1\rangle)$ $(i=1,\ldots,n;$ the bar stands for complex conjugation) and the sum is over the elements of the permutation group $\mathcal{S}_n$ of n objects. In Eq. (1), the vector $|\eta_{\sigma(1)},\ldots,\eta_{\sigma(n)}\rangle$ stands for the tensor product $|\eta_{\sigma(1)}\rangle\otimes\cdots\otimes|\eta_{\sigma(n)}\rangle$. The totally symmetric n-qubit states can be also formulated in Dicke representation. The symmetric Dicke states with k excitations are defined by[35]

$$|n,k\rangle = \sqrt{\frac{k!(n-k)!}{n!}} \sum_{\sigma \in \mathcal{S}_n} |\underbrace{0,\ldots,0}_{n-k},\underbrace{1,\ldots,1}_{k}\rangle\,, \tag{2}$$

which generate an orthonormal basis of the symmetric Hilbert subspace of dimension $(n+1)$. Therefore, permutation invariance, in symmetric multiqubit states, implies a restriction to $(n+1)$-dimensional subspace from the entire 2^n-dimensional Hilbert space. The Dicke states (2) constitute a special subset of the symmetric multiqubit states (1) corresponding to the situation where the first k qubits are such that $\eta_i = 0$ for $i = 0,1,\ldots,k$ and the remaining qubits are in the states $|\eta_i = 1\rangle$ with $i = k+1,\ldots,n$. Any symmetric state $|\psi_s\rangle$ (1) can be expanded in terms of Dicke states (2) as follows:

$$|\psi_s\rangle = \frac{1}{n!}\sum_{k=0}^{n} c_k |n,k\rangle\,, \tag{3}$$

where c_k $(k = 0,\ldots,n)$ stand for the complex expansion coefficients. In particular, when the qubits are all identical ($\eta_i = \eta$ for all qubits), it is simply verified that the coefficients c_k are given by

$$c_k = n!\sqrt{\frac{n!}{k!(n-k)!}}\frac{\eta^k}{(1+\eta\bar{\eta})^{\frac{n}{2}}} \tag{4}$$

and the symmetric multiqubit states (1) write

$$|\psi_s\rangle := |n,\eta\rangle = (1+\eta\bar{\eta})^{-\frac{n}{2}} \sum_{k=0}^{n} \sqrt{\frac{n!}{k!(n-k)!}}\eta^k |n,k\rangle\,, \tag{5}$$

which are exactly the $j = n/2$-spin coherent states (for more details see for instance Ref. 36). In particular, the state $|n,\eta\rangle$ can be identified for $n = 1$ with spin-1/2 coherent state with $|0\rangle \equiv |(1/2),-(1/2)\rangle$ and $|1\rangle \equiv |(1/2),+(1/2)\rangle$.

2.2. *Multiqubit "Schrödinger cat" states*

The prototypical multiqubit "Schrödinger cat" states, we consider in this work, are defined as a balanced superpositions of the n-qubit states $|n,\eta\rangle$ and $|n,-\eta\rangle$ given by (5). They write

$$|\eta,n,m\rangle = \mathcal{N}(|n,\eta\rangle + e^{im\pi}|n,-\eta\rangle)\,, \tag{6}$$

where

$$|n,\pm\eta\rangle = |\pm\eta\rangle \otimes |\pm\eta\rangle \cdots \otimes |\pm\eta\rangle$$

and the integer $m \in \mathbb{Z}$ takes the values $m = 0 \pmod 2$ and $m = 1 \pmod 2$. The normalization factor $\mathcal{N}$ is

$$\mathcal{N} = \left[2 + 2p^n \cos m\pi\right]^{-1/2},$$

where p denotes the overlap between the states $|\eta\rangle$ and $|-\eta\rangle$. It is given by

$$p = \langle\eta|-\eta\rangle = \frac{1-\bar{\eta}\eta}{1+\bar{\eta}\eta}\,. \tag{7}$$

Experimental creation of cat states comprising multiple particles was reported in the literature.[37,38] Due to their experimental implementation, "Schrödinger cat" states are expected to be an useful resource for quantum computing as well as quantum communications. Also, in view of their mathematical elegance, multiqubit states obeying exchange symmetry offer drastic simplification in investigating various aspects of quantum correlations in particular the geometric measure of quantum discord as we shall discuss in the present work. Furthermore, the multiqubit symmetric states (6) include Greenberger–Horne–Zeilinger GHZ,[39] W (Ref. 40) and Dicke states.[35] The multiqubits states $|n,\eta,0\rangle$ ($m = 0 \bmod 2$) and $|n,\eta,1\rangle$ ($m = 1 \bmod 2$) behave like a multipartite state of GHZ-type[39] in the limiting case $p \to 0$. Indeed, the states $|\eta\rangle$ and $|-\eta\rangle$ approach orthogonality and an orthogonal basis can be defined such that $|\mathbf{0}\rangle \equiv |\eta\rangle$ and $|\mathbf{1}\rangle \equiv |-\eta\rangle$. Thus, the state $|n,\eta,m\rangle$ becomes of GHZ-type:

$$|\eta,n,m\rangle \sim |\mathrm{GHZ}\rangle_n = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle\otimes|\mathbf{0}\rangle\otimes\cdots\otimes|\mathbf{0}\rangle + e^{im\pi}|\mathbf{1}\rangle\otimes|\mathbf{1}\rangle\otimes\cdots\otimes|\mathbf{1}\rangle)\,. \tag{8}$$

Also, in the special situation where the overlap p tends to unity ($p \to 1$ or $\eta \to 0$), the state $|\eta,n,m = 0 \pmod 2\rangle$ (6) reduces to ground state of a collection of n qubits:

$$|0,n,0 \pmod 2\rangle \sim |0\rangle\otimes|0\rangle\otimes\cdots\otimes|0\rangle \tag{9}$$

and it is simple to check that the state $|\eta,0,1 \pmod 2\rangle$ becomes a multipartite state of W-type[40]

$$\begin{aligned}|0,n,1 \pmod 2\rangle &\sim |\mathrm{W}\rangle_n \\ &= \frac{1}{\sqrt{n}}(|1\rangle\otimes|0\rangle\otimes\cdots\otimes|0\rangle + |0\rangle\otimes|1\rangle\otimes\cdots\otimes|0\rangle + \cdots + |0\rangle\otimes|0\rangle\otimes\cdots\otimes|1\rangle)\,.\end{aligned} \tag{10}$$

It is clear that the Schrödinger cat states $|\eta,n,m = 0 \pmod 2\rangle$ include the GHZ_n states ($p \to 0$). In other hand, the states $|\eta,n,m = 1 \pmod 2\rangle$, constitute an interpolation between two special classes of multiqubits states: $|\mathrm{GHZ}\rangle_n$ type corresponding to $p \to 0$ and states of $|\mathrm{W}\rangle_n$ type obtained in the special case where $p \to 1$.

3. Multipartite Quantum Correlations

The structure of multipartite correlations within multiqubit quantum systems is a challenging and daunting task. With the growth of number of qubits, there are numerous ways in splitting the entire system to characterize how the particles are correlated. Obviously, the bipartite splitting of the whole system is not sufficient to capture the essential of quantum correlation existing in a multiqubit system. However, it must be noticed that the pairwise decomposition of total correlation offers a good alternative to evaluate the amount of all correlations existing in a multipartite system. In this paper, we approach the problem of analyzing n-qubit correlation using only bipartite measures. Toward this end, we consider first the correlation between one qubit with the remaining $(n-1)$ qubits in the state (6). Thus, the pure density matrix of the symmetric n-qubit system writes

$$\rho_n \equiv |\eta, n, m\rangle\langle\eta, n, m| := \rho_{1|23\cdots n}\,.$$

Furthermore, after removing $k = 1, 2, \ldots, n-2$ particles from the n-qubit system, the reduced density matrix ρ_{n-k} can be bipartitioned in two subsystems, one comprises of one qubit and the remaining $(n-k-1)$ qubits are contained in the second subsystem. In this manner, a bipartite measure characterize the pairwise correlation between the two subsystems. This offers a reasonable scheme to characterize the total amount of quantum correlation defined as the sum of the quantum correlations for all possible bipartitions.[41–43]

In this paper, we shall employ this picture to estimate the geometric measure of quantum discord (D_g) in the symmetric multiqubit system of the form (6). We give a detailed analysis for two-qubit and three-qubit subsystems. From these two specific cases, we give a general algorithm to determine recursively the pairwise quantum discord in a reduced density describing k-qubit system.

3.1. *Two-qubit states*

We begin with the two-qubit case. The tools we introduce are useful when extending the size of the system to encompass more qubits. We first consider the two-qubit states extracted from the state (6) by tracing out $(n-2)$ qubits. Since the n qubits are all identical, we obtain $n(n-1)/2$ identical density matrices. They are given by

$$\begin{aligned}\rho_{12} = \mathcal{N}^2[&|\eta,\eta\rangle\langle\eta,\eta| + e^{im\pi}q_2|-\eta,-\eta\rangle\langle\eta,\eta|\\ &+ e^{-im\pi}q_2|\eta,\eta\rangle\langle-\eta,-\eta| + |-\eta,-\eta\rangle\langle-\eta\,,-\eta|]\,,\end{aligned} \tag{11}$$

where q_2 is defined by $q_s = p^{n-s}$ with $s = 2$. The state (11) can be alternatively written as

$$\rho_{12} = \frac{1}{2}(1+p^{n-2})\frac{\mathcal{N}^2}{{\mathcal{N}_{2+}}^2}|\eta\rangle_2\,{}_2\langle\eta| + \frac{1}{2}\,(1-p^{n-2})\frac{\mathcal{N}^2}{{\mathcal{N}_{2-}}^2}Z|\eta\rangle_2\,{}_2\langle\eta|Z\,, \tag{12}$$

with

$$|\eta\rangle_2 = \mathcal{N}_{2+}(|\eta,\eta\rangle + e^{im\pi}|-\eta,-\eta\rangle) \quad \text{and} \quad Z|\eta\rangle_2 = \mathcal{N}_{2-}(|\eta,\eta\rangle - e^{im\pi}|-\eta,-\eta\rangle)\,.$$

The normalization factors are defined by

$$\mathcal{N}_{s\pm}^{-2} = 2(1 \pm p^s\cos m\pi)\,,$$

for $s = 2$. In the computational base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the density matrix ρ_{12} has the form of the alphabet X. Indeed, it is represented by

$$\rho_{12} = 2\mathcal{N}^2 \begin{pmatrix} q_{2+}a_+^4 & 0 & 0 & q_{2+}a_+^2 a_-^2 \\ 0 & q_{2-}a_+^2 a_-^2 & q_{2-}a_+^2 a_-^2 & 0 \\ 0 & q_{2-}a_+^2 a_-^2 & q_{2-}a_+^2 a_-^2 & 0 \\ q_{2+}a_+^2 a_-^2 & 0 & 0 & q_{2+}a_-^4 \end{pmatrix}, \tag{13}$$

where

$$a_\pm = \frac{\sqrt{1 \pm p}}{\sqrt{2}} \quad \text{and} \quad q_{s\pm} = 1 \pm q_s \cos m\pi \,.$$

The state ρ_{12} can be also written as

$$\rho_{12} = \sum_{k,l=0,1} \rho^{kl} \otimes |k\rangle\langle l| \,. \tag{14}$$

This form is suitable to establish a relation between the Bloch components of the 2×2 matrices ρ^{kl} and the correlation matrix elements associated with the two-qubit state ρ_{12}. In Eq. (14), the matrices ρ^{ij} writes in Bloch representation as

$$\rho^{00} = \frac{1}{2}(T_0^{00}\sigma_0 + T_3^{00}\sigma_3)\,, \quad \rho^{11} = \frac{1}{2}(T_0^{11}\sigma_0 + T_3^{11}\sigma_3) \tag{15}$$

and

$$\rho^{01} = \frac{1}{2}(T_1^{01}\sigma_1 + T_2^{01}\sigma_2)\,, \quad \rho^{10} = \frac{1}{2}(T_1^{10}\sigma_1 + T_2^{10}\sigma_2)\,, \tag{16}$$

where the Bloch components T_α^{kl} ($\alpha = 0,1,2,3$) are

$$T_0^{kk} = \mathcal{N}^2(1 + (-)^k p)(1 + (-)^k p^{n-1}\cos m\pi)\,,$$

$$T_3^{kk} = \mathcal{N}^2(1 + (-)^k p)(1 + (-)^k p^{n-2}\cos m\pi)\,,$$

for $k = 0,1$ and

$$T_1^{01} = T_1^{10} = \mathcal{N}^2(1 - p^2), \quad T_2^{01} = -T_2^{10} = i\mathcal{N}^2(1 - p^2)p^{n-2}\cos m\pi.$$

Reporting (15) and (16) in (14), one gets

$$\rho_{12} = \sum_{\alpha\beta} T_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta\,, \tag{17}$$

where the nonvanishing matrix elements $T_{\alpha\beta}$ ($\alpha, \beta = 0,1,2,3$) are given by

$$\begin{gathered} T_{\alpha 0} = T_\alpha^{00} + T_\alpha^{11}, \quad \text{for} \quad \alpha = 0,3\,, \quad T_{\alpha 1} = T_\alpha^{01} + T_\alpha^{10}, \quad \text{for} \quad \alpha = 1\,, \\ T_{\alpha 2} = iT_\alpha^{01} - iT_\alpha^{10}, \quad \text{for} \quad \alpha = 2\,, \quad T_{\alpha 3} = T_\alpha^{00} - T_\alpha^{11}, \quad \text{for} \quad \alpha = 0,3\,, \end{gathered} \tag{18}$$

which gives

$$\begin{gathered} T_{00} = 1\,, \quad T_{11} = 2\mathcal{N}^2(1 - p^2)\,, \quad T_{22} = -2\mathcal{N}^2(1 - p^2)p^{n-2}\cos m\pi\,, \\ T_{33} = 2\mathcal{N}^2(p^2 + p^{n-2}\cos m\pi)\,, \quad T_{03} = T_{30} = 2\mathcal{N}^2(p + p^{n-1}\cos m\pi)\,. \end{gathered} \tag{19}$$

The expressions (18) establish the relations between the Bloch components associated with one-qubit states (15) and (16) and the two-qubit Fano–Bloch tensor elements T_α^{kl} occurring in the two-qubit density ρ_{12} (17). This result is generalizable to more qubits. This issue is discussed in what follows.

3.2. *Three-qubit states*

The three-qubit states is extracted from the whole state (6) by removing $(n-3)$ qubits by the usual trace procedure. In this case, one obtains $n(n-1)(n-2)/3!$ density matrices which are all identical. Explicitly, they are given by

$$\begin{aligned}\rho_{123} = \mathcal{N}^2 \big[&|\eta,\eta,\eta\rangle\langle\eta,\eta,\eta| + e^{im\pi} q_3 |-\eta,-\eta,-\eta\rangle\langle\eta,\eta,\eta| \\ &+ e^{-im\pi} q_3 |\eta,\eta,\eta\rangle\langle-\eta,-\eta,-\eta| + |-\eta,-\eta,-\eta\rangle\langle-\eta,-\eta,-\eta| \big]\,, \end{aligned} \tag{20}$$

where $q_3 = p^{n-3}$. Analogously to the previous case, we write the mixed three-qubit state ρ_{123} in a more compact form as follows:

$$\rho_{123} = \frac{1}{2}(1+p^{n-3})\frac{\mathcal{N}^2}{\mathcal{N}_{3+}^2}|\eta\rangle_3\,{}_3\langle\eta| + \frac{1}{2}(1-p^{n-3})\frac{\mathcal{N}^2}{\mathcal{N}_{3-}^2} Z|\eta\rangle_3\,{}_3\langle\eta|Z\,, \tag{21}$$

where

$$|\eta\rangle_3 = \mathcal{N}_{3+}(|\eta,\eta,\eta\rangle + e^{im\pi}|-\eta,-\eta,-\eta\rangle)\,,$$

$$Z|\eta\rangle_3 = \mathcal{N}_{3-}(|\eta,\eta,\eta\rangle - e^{im\pi}|-\eta,-\eta,-\eta\rangle)\,,$$

with the normalization factors $\mathcal{N}_{3\pm}$ given by

$$\mathcal{N}_{3\pm}^{-2} = 2(1 \pm p^3 \cos m\pi)\,.$$

In the computational base $\{|000\rangle, |010\rangle, |100\rangle, |110\rangle, |001\rangle, |011\rangle, |101\rangle, |111\rangle\}$, the state ρ_{123} takes the matrix form:

$$\begin{aligned}&\frac{\rho_{123}}{2\mathcal{N}^2} \\ &= \begin{pmatrix} q_{+3}a_+^6 & 0 & 0 & q_{+3}a_+^4a_-^2 & 0 & q_{+3}a_+^4a_-^2 & q_{+3}a_+^4a_-^2 & 0 \\ 0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 & q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\ 0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 & q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\ q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 & 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\ 0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 & q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\ q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 & 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\ q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 & 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\ 0 & q_{-3}a_+^2a_-^4 & q_{-3}a_+^2a_-^4 & 0 & q_{-3}a_+^2a_-^4 & 0 & 0 & q_{-3}a_-^6 \end{pmatrix}. \end{aligned} \tag{22}$$

The state (22) can be also rewritten as

$$\rho_{123} = \sum_{k,l=0,1} \rho^{kl} \otimes |k\rangle\langle l|\,, \tag{23}$$

where $|k\rangle$, $|l\rangle$ are related to the qubit three. The two-qubit density matrices ρ^{kk} (for $k = 0, 1$) writes, in the computational basis spanned by $\{|0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2, |1\rangle_1 \otimes |0\rangle_2, |1\rangle_1 \otimes |1\rangle_2\}$, as

$$\rho^{00} = 2\mathcal{N}^2 \begin{pmatrix} q_{+3}a_+^6 & 0 & 0 & q_{+3}a_+^4 a_-^2 \\ 0 & q_{-3}a_+^4 a_-^2 & q_{-3}a_+^4 a_-^2 & 0 \\ 0 & q_{-3}a_+^4 a_-^2 & q_{-3}a_+^4 a_-^2 & 0 \\ q_{+3}a_+^4 a_-^2 & 0 & 0 & q_{+3}a_+^2 a_-^4 \end{pmatrix} \tag{24}$$

and

$$\rho^{11} = 2\mathcal{N}^2 \begin{pmatrix} q_{-3}a_+^4 a_-^2 & 0 & 0 & q_{-3}a_+^2 a_-^4 \\ 0 & q_{+3}a_+^2 a_-^4 & q_{+3}a_+^2 a_-^4 & 0 \\ 0 & q_{+3}a_+^2 a_-^4 & q_{+3}a_+^2 a_-^4 & 0 \\ q_{-3}a_+^2 a_-^4 & 0 & 0 & q_{-3}a_-^6 \end{pmatrix}. \tag{25}$$

For $(k = 0, l = 1)$ and $(k = 1, l = 0)$, we have respectively

$$\rho^{01} = 2\mathcal{N}^2 \begin{pmatrix} 0 & q_{+3}a_+^4 a_-^2 & q_{+3}a_+^4 a_-^2 & 0 \\ q_{-3}a_+^4 a_-^2 & 0 & 0 & q_{-3}a_+^2 a_-^4 \\ q_{-3}a_+^4 a_-^2 & 0 & 0 & q_{-3}a_+^2 a_-^4 \\ 0 & q_{+3}a_+^2 a_-^4 & q_{+3}a_+^2 a_-^4 & 0 \end{pmatrix} \tag{26}$$

and

$$\rho^{10} = 2\mathcal{N}^2 \begin{pmatrix} 0 & q_{-3}a_+^4 a_-^2 & q_{-3}a_+^4 a_-^2 & 0 \\ q_{+3}a_+^4 a_-^2 & 0 & 0 & q_{+3}a_+^2 a_-^4 \\ q_{+3}a_+^4 a_-^2 & 0 & 0 & q_{+3}a_+^2 a_-^4 \\ 0 & q_{-3}a_+^2 a_-^4 & q_{-3}a_+^2 a_-^4 & 0 \end{pmatrix}. \tag{27}$$

The Fano–Bloch representation of the matrices ρ^{kk}, given by (24) and (25), takes the form:

$$\rho^{kk} = \frac{1}{4}\sum_{\alpha\beta} T_{\alpha\beta}^{kk}\sigma_\alpha \otimes \sigma_\beta, \tag{28}$$

where $\alpha, \beta = 0, 1, 2, 3$ and the correlation matrix elements $T_{\alpha\beta}^{kk}$ are given by

$$T_{\alpha\beta}^{kk} = \mathrm{Tr}(\rho^{kk}\ \sigma_\alpha \otimes \sigma_\beta)\,.$$

The explicit expressions of the nonvanishing contributions are

$$\begin{aligned}
T_{00}^{kk} &= 1\,,\\
T_{30}^{kk} &= T_{03}^{kk} = \frac{p}{2}(1+(-)^k p)\frac{1+(-)^k p^{n-3}\cos m\pi}{1+p^n\cos m\pi}\,,\\
T_{11}^{kk} &= \frac{1}{2}(1+(-)^k p)\frac{1-p^2}{1+p^n\cos m\pi}\,,\\
T_{22}^{kk} &= -\frac{1}{2}(1+(-)^k p)\frac{(1-p^2)p^{n-3}\cos m\pi}{1+p^n\cos m\pi}\,,\\
T_{33}^{kk} &= \frac{1}{2}\ (1+(-)^k p)\frac{p^2+(-)^k p^{n-3}\cos m\pi}{1+p^n\cos m\pi}\,.
\end{aligned} \tag{29}$$

Similarly, for the two-qubit states ρ^{kl} $(k \neq l)$ given by (26) and (27), the Fano–Bloch representation writes

$$\rho^{kl} = \frac{1}{4}\sum_{\alpha\beta} T_{\alpha\beta}^{kl}\sigma_\alpha \otimes \sigma_\beta\,, \tag{30}$$

where the nonzero matrix elements $T_{\alpha\beta}^{kl}$ are given by

$$\begin{aligned}
T_{01}^{kl} &= T_{10}^{kl} = \frac{1}{2}\frac{1-p^2}{1+p^n\cos m\pi}\,,\\
T_{02}^{kl} &= T_{20}^{kl} = (-)^k\frac{i}{2}\frac{p(1-p^2)}{1+p^n\cos m\pi}\,,\\
T_{13}^{kl} &= T_{31}^{kl} = \frac{1}{2}\frac{(1-p^2)p^{n-2}\cos m\pi}{1+p^n\cos m\pi}\,,\\
T_{23}^{kl} &= T_{32}^{kl} = (-)^k\frac{i}{2}\frac{(1-p^2)p^{n-2}\cos m\pi}{1+p^n\cos m\pi}\,.
\end{aligned} \tag{31}$$

Using (23), the three-qubit state ρ_{123} expands as

$$\rho_{123} = \frac{1}{2}\left[(\rho^{00}+\rho^{11})\otimes\sigma_0 + (\rho^{00}-\rho^{11})\otimes\sigma_3 + (\rho^{01}+\rho^{10})\otimes\sigma_1 + i(\rho^{01}-\rho^{10})\otimes\sigma_2\right]. \tag{32}$$

Inserting (28) and (30) in Eq. (32) and using the results (29) and (31), one gets

$$\begin{aligned}
\rho_{123} = \frac{1}{8}\sum_{\alpha\beta}\big[&T_{\alpha\beta 0}\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_0 + T_{\alpha\beta 1}\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_1 + T_{\alpha\beta 2}\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_2\\
&+ T_{\alpha\beta 3}\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_3\big]\,,
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
T_{\alpha\beta 0} &= T_{\alpha\beta}^{++} = T_{\alpha\beta}^{00} + T_{\alpha\beta}^{11}\,,\\
T_{\alpha\beta 3} &= T_{\alpha\beta}^{--} = T_{\alpha\beta}^{00} - T_{\alpha\beta}^{11}\,,
\end{aligned} \tag{34}$$

with $\alpha\beta = 00, 03, 30, 11, 22, 33$ [*cf.* (29)], and

$$T_{\alpha\beta 1} = T^{+-}_{\alpha\beta} = T^{01}_{\alpha\beta} + T^{10}_{\alpha\beta}\,,$$
$$T_{\alpha\beta 2} = T^{-+}_{\alpha\beta} = iT^{01}_{\alpha\beta} - iT^{10}_{\alpha\beta}\,, \tag{35}$$

with $\alpha\beta = 01, 02, 10, 20, 13, 23, 31, 32$ (*cf.* (31)). Reporting (29) and (31) in the expressions (34) and (35), one obtains the 32 nonvanishing correlation matrix elements $T_{\alpha\beta\gamma}$ corresponding to the three-qubit state ρ_{123}. Subsequently, the recursive relations (34) and (35) offer a nice tool to determine the correlation elements $T_{\alpha\beta\gamma}$ in terms of those associated with the two-qubit density matrices ρ^{kk} and ρ^{kl} given, respectively, by (28) and (30). Clearly, along the same line of reasoning, the recursive relation obtained for two and three qubits are ready to be extended to an arbitrary k-qubit state.

3.3. *k-qubit states*

A mixed k-qubit state ($k = 2, 3, \ldots, n$) is obtained by tracing out $(n-k)$ qubits from the state (6). It is given by

$$\begin{aligned}\rho_{123\cdots k} = \mathcal{N}^2\big[&|\eta,\eta,\ldots,\eta\rangle\langle\eta,\eta,\ldots,\eta| + e^{im\pi}q_k|-\eta,-\eta,\ldots,-\eta\rangle\langle\eta,\eta,\ldots,\eta| \\ &+ e^{-im\pi}q_k|\eta,\eta,\ldots,\eta\rangle\langle-\eta,-\eta,\ldots,-\eta| \\ &+ |-\eta,-\eta,\ldots,-\eta\rangle\langle-\eta,-\eta,\ldots-\eta|\big]\,,\end{aligned} \tag{36}$$

where $q_k = p^{n-k}$. The reduced density matrix $\rho_{123\cdots k}$ is of rank two. Indeed, the state (36) rewrites

$$\rho_{123\cdots k} = \frac{1}{2}(1+p^{n-k})\frac{\mathcal{N}^2}{\mathcal{N}^2_{k+}}|\eta\rangle_k\,{}_k\langle\eta| + \frac{1}{2}(1-p^{n-k})\frac{\mathcal{N}^2}{\mathcal{N}^2_{k-}}Z|\eta\rangle_k\,{}_k\langle\eta|Z\,, \tag{37}$$

where

$$|\eta\rangle_k = \mathcal{N}_{k+}(|\eta,\eta,\ldots,\eta\rangle + e^{im\pi}|-\eta,-\eta,\ldots,-\eta\rangle)\,,$$
$$Z|\eta\rangle_k = \mathcal{N}_{k-}(|\eta,\eta,\ldots,\eta\rangle - e^{im\pi}|-\eta,-\eta,\ldots,-\eta\rangle)$$

and the normalization factors $\mathcal{N}_{k\pm}$ are given by

$$\mathcal{N}^{-2}_{k\pm} = 2(1 \pm p^k\cos m\pi)\,.$$

The cyclic operator Z is now defined by

$$Z|\eta,\eta,\ldots,\eta\rangle = |\eta,\eta,\ldots,\eta\rangle \quad Z|-\eta,-\eta,\ldots,-\eta\rangle = -|-\eta,-\eta,\ldots,-\eta\rangle\,.$$

Using (36), it is simple to check that the k-qubit state $\rho_{123\cdots k}$ can be expressed in terms of states comprising $(k-1)$ qubits. The state $\rho_{123\cdots k}$ (36) can be written also as

$$\rho_{123\cdots k} = \sum_{rs=1,2}\rho^{rs}_{12\cdots(k-1)}\otimes|r\rangle\langle s|\,, \tag{38}$$

where

$$\rho^{rs}_{12\cdots(k-1)} \equiv \rho^{rs} = a_+^{2-r-s} a_-^{r+s} \left[\frac{1}{2}(1+p^{n-k}) \frac{\mathcal{N}^2}{\mathcal{N}^2_{(k-1)+}} Z^r |\eta\rangle_{(k-1)(k-1)} \langle \eta | Z^s \right.$$
$$\left. + \frac{1}{2}(1-p^{n-k}) \frac{\mathcal{N}^2}{\mathcal{N}^2_{(k-1)-}} Z^{r+1} |\eta\rangle_{(k-1)(k-1)} \langle \eta | Z^{s+1} \right]. \tag{39}$$

Explicitly, the k-qubit matrix (38) writes

$$\rho_{123\cdots k} = \frac{1}{2}(\rho^{00}+\rho^{11}) \otimes \sigma_0 + \frac{1}{2}(\rho^{01}+\rho^{10}) \otimes \sigma_1$$
$$+ \frac{i}{2}(\rho^{01}-\rho^{10}) \otimes \sigma_2 + \frac{1}{2}(\rho^{00}-\rho^{11}) \otimes \sigma_3 \tag{40}$$

and the $(k-1)$-qubit states ρ^{rs} can be expanded, in Fano–Bloch representation, as

$$\rho^{rs} = \frac{1}{2^{k-1}} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}} T^{rs}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} \sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \cdots \otimes \sigma_{\alpha_{k-1}}. \tag{41}$$

Hence, reporting (41) in (38), the k-qubit state $\rho_{123\cdots k}$ takes the form:

$$\rho_{123\cdots k} = \frac{1}{2^k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{k-1},\alpha_k} T_{\alpha_1\alpha_2\cdots\alpha_{k-1}\alpha_k} \sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \cdots \otimes \sigma_{\alpha_{k-1}} \otimes \sigma_{\alpha_k}, \tag{42}$$

where the correlation matrix elements $T_{\alpha_1\alpha_2\cdots\alpha_{k-1}\alpha_k}$ expressed in terms of the correlation coefficients occurring in (41) as

$$\begin{aligned} T_{\alpha_1\alpha_2\cdots\alpha_{k-1}0} &= T^{00}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} + T^{11}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}, \\ T_{\alpha_1\alpha_2\cdots\alpha_{k-1}3} &= T^{00}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} - T^{11}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}, \\ T_{\alpha_1\alpha_2\cdots\alpha_{k-1}1} &= T^{01}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} + T^{10}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}, \\ T_{\alpha_1\alpha_2\cdots\alpha_{k-1}2} &= iT^{01}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} - iT^{10}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} \end{aligned} \tag{43}$$

and we have the relations between the correlation matrix elements of k and $(k-1)$-qubit states. In this picture the correlation matrix elements associated with a k-qubit state can be recursively expressed in terms of the ones involving two qubits. It is simply verified that the relations (43) reduce to (18) for $k=2$ and to (34), (35) for $k=3$. To illustrate the algorithm in deriving relations of type (43), we consider the case of four qubits. In this situation, the density matrix (36) becomes

$$\rho_{1234} = \mathcal{N}^2 \big[|\eta,\eta,\eta,\eta\rangle\langle\eta,\eta,\eta,\eta| + e^{im\pi} q_4 |-\eta,-\eta,-\eta,-\eta\rangle\langle\eta,\eta,\eta,\eta|$$
$$+ e^{-im\pi} q_4 |\eta,\eta,\eta,\eta\rangle\langle-\eta,-\eta,-\eta,-\eta| + |-\eta,-\eta,-\eta,-\eta\rangle\langle-\eta,-\eta,-\eta-\eta| \big] \tag{44}$$

and the expression (38) gives

$$\rho_{1234} = \rho^{00}_{123} \otimes |0\rangle\langle 0| + \rho^{01}_{123} \otimes |0\rangle\langle 1| + \rho^{10}_{123} \otimes |1\rangle\langle 0| + \rho^{11}_{123} \otimes |1\rangle\langle 1|, \tag{45}$$

where the three-qubit states ρ_{123}^{00}, ρ_{123}^{01}, ρ_{123}^{10} and ρ_{123}^{11} are given in the usual computational basis as

$$\frac{\rho_{123}^{00}}{2\mathcal{N}^2} = \begin{pmatrix} q_{+4}a_+^8 & 0 & 0 & q_{+4}a_+^6a_-^2 & 0 & q_{+4}a_+^6a_-^2 & q_{+4}a_+^6a_-^2 & 0 \\ 0 & q_{-4}a_+^6a_-^2 & q_{-4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\ 0 & q_{-4}a_+^6a_-^2 & q_{-4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\ q_{+4}a_+^6a_-^2 & 0 & 0 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\ 0 & q_{-4}a_+^6a_-^2 & q_{-4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\ q_{+4}a_+^6a_-^2 & 0 & 0 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\ q_{+4}a_+^6a_-^2 & 0 & 0 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\ 0 & q_{-4}a_+^4a_-^4 & q_{-4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 \end{pmatrix}, \tag{46}$$

$$\frac{\rho_{123}^{11}}{2\mathcal{N}^2} = \begin{pmatrix} q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & q_{-4}a_+^4a_-^4 & 0 \\ 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\ 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\ q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \\ 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\ q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \\ q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \\ 0 & q_{+4}a_+^2a_-^6 & q_{+4}a_+^2a_-^6 & 0 & q_{+4}a_+^2a_-^6 & 0 & 0 & q_{+4}a_-^8 \end{pmatrix}, \tag{47}$$

$$\frac{\rho_{123}^{01}}{2\mathcal{N}^2} = \begin{pmatrix} 0 & q_{+4}a_+^6a_-^2 & q_{+4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\ q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\ q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\ 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 \\ q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & q_{-4}a_+^4a_-^4 & 0 \\ 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\ 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\ q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \end{pmatrix} \tag{48}$$

and

$$\rho_{123}^{10} = \left(\rho_{123}^{01}\right)^t. \tag{49}$$

It is clear that with increasing the qubits number, complicated analytical computation emerges especially in computing the quantum correlations. However, the recursive algorithm presented above, offers an alternative way for symmetric multi-qubit (6), to reduce the complexity in determining analytical evaluation of geometric discord. The expression (45) allows us to express the correlations factors $T_{\alpha_1\alpha_2\alpha_3\alpha_4}$ in terms of those corresponding to three-qubit density matrices ρ_{123}^{00}, ρ_{123}^{01}, ρ_{123}^{10} and ρ_{123}^{11}. Indeed, the state ρ_{1234} writes in the Fano–Bloch representation as

$$\rho_{1234} = \frac{1}{2^4} \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} T_{\alpha_1\alpha_2\alpha_3\alpha_4} \sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \sigma_{\alpha_3} \otimes \sigma_{\alpha_4} \tag{50}$$

and by reequating (45) as

$$\begin{aligned}\rho_{1234} = {} & \frac{1}{2}(\rho_{123}^{00} + \rho_{123}^{11}) \otimes \sigma_0 + \frac{1}{2}(\rho_{123}^{01} + \rho_{123}^{10}) \otimes \sigma_1 + \frac{i}{2}(\rho_{123}^{01} - \rho_{123}^{10}) \otimes \sigma_2 \\ & + \frac{1}{2}(\rho_{123}^{00} - \rho_{123}^{00}) \otimes \sigma_3 \,,\end{aligned} \tag{51}$$

it is simple to see that

$$\begin{aligned} T_{\alpha_1\alpha_2\alpha_3 0} &= T^{00}_{\alpha_1\alpha_2\alpha_3} + T^{11}_{\alpha_1\alpha_2\alpha_3} \,, \\ T_{\alpha_1\alpha_2\alpha_3 1} &= T^{01}_{\alpha_1\alpha_2\alpha_3} + T^{10}_{\alpha_1\alpha_2\alpha_3} \,, \\ T_{\alpha_1\alpha_2\alpha_3 2} &= iT^{01}_{\alpha_1\alpha_2\alpha_3} - iT^{10}_{\alpha_1\alpha_2\alpha_3} \,, \\ T_{\alpha_1\alpha_2\alpha_3 3} &= T^{00}_{\alpha_1\alpha_2\alpha_3} - T^{11}_{\alpha_1\alpha_2\alpha_3} \,, \end{aligned} \tag{52}$$

where the quantities $T^{kl}_{\alpha_1,\alpha_2,\alpha_3}$, defined so that

$$\rho_{123}^{kl} = \frac{1}{2^3} \sum_{\alpha_1,\alpha_2,\alpha_3} T^{kl}_{\alpha_1\alpha_2\alpha_3} \sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \sigma_{\alpha_3} \,, \tag{53}$$

can be obtained easily following the method developed above for three- and two-qubit states. It follows that the nonvanishing elements $T_{\alpha_1\alpha_2\alpha_3\alpha_4}$ are those with indices $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ belonging to the following set of quadruples:

$$\begin{gathered} \{00,11,22,33,03,30\} \times \{0,3\} \times \{0,3\} \,, \\ \{00,11,22,33,03,30\} \times \{1,2\} \times \{1,2\} \,, \\ \{01,10,20,02,13,31,23,32\} \times \{1,2\} \times \{0,3\} \,, \\ \{01,10,20,02,13,31,23,32\} \times \{0,3\} \times \{1,2\}\} \,. \end{gathered}$$

Finally, we stress the usefulness of the recursive approach, discussed in this section, in determining the Fano–Bloch components for an arbitrary k-qubit state in terms of those involving $(k-1)$-qubits. This gives a simple way to specify the correlation matrix elements for k-qubit state in terms of ones associated with two-qubit subsystems. In this picture, for the symmetric multiqubit states (6), considerable simplification arises in establishing such recursive relations and subsequently simplify drastically the evaluation of pairwise geometric quantum discord.

4. Geometric Measure of Quantum Discord and Classical States

We now face the question of determining the explicit form of geometric discord between a qubit and a second parity of dimension 2^{k-1} in the k-qubit mixed state (36). For this end, we must first find the expression of closest classical states to the states of type (36) when the distance is measured by Hilbert–Schmidt trace. We shall follow the procedure developed in Ref. 15 for a two-qubit system.

4.1. *Two-qubit states*

For the two-qubit state (17) which rewrites

$$\rho_{12} = \frac{1}{4}\big[\sigma_0 \otimes \sigma_0 + T_{30}\sigma_3 \otimes \sigma_0 + T_{03}\sigma_0 \otimes \sigma_3 + T_{11}\sigma_1 \otimes \sigma_1 + T_{22}\sigma_2 \otimes \sigma_2 + T_{33}\sigma_3 \otimes \sigma_3\big]\,, \tag{54}$$

the zero discord or classical states are given by

$$\chi_{12} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^2 + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^2\,, \tag{55}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit one and ρ_i^2 $(i = 1, 2)$ are the reduced density matrices attached to the second qubit. It can be written also as

$$\chi_{12} = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} te_i\sigma_i \otimes \sigma_0 + \sum_{i=1}^{3}(s_+)_i\sigma_0 \otimes \sigma_i + \sum_{i,j=1}^{3} e_i(s_-)_j\sigma_i \otimes \sigma_j\right], \tag{56}$$

where

$$t = p_1 - p_2\,, \quad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle\,, \quad (s_\pm)_j = \mathrm{Tr}\big((p_1\rho_1^2 \pm p_2\rho_2^2)\sigma_j\big)\,.$$

The distance between the density matrix ρ_{12} (54) and the classical state χ_{12} (56), as measured by Hilbert–Schmidt norm, is

$$\|\rho_{12} - \chi_{12}\|^2 = \frac{1}{4}\left[(t^2 - 2te_3T_{30} + T_{30}^2) + \sum_{i=1}^{3}(T_{0i} - (s_+)_i)^2 + \sum_{i,j=1}^{3}(T_{ij} - e_i(s_-)_j)^2\right]. \tag{57}$$

The minimal distance is obtained by minimizing the Hilbert–Schmidt norm (57) with respect to the parameters t, $(s_+)_i$ and $(s_-)_i$. This gives

$$\begin{gathered} t = e_3 T_{30}\,, \\ (s_+)_1 = 0\,, \quad (s_+)_2 = 0\,, \quad (s_+)_3 = T_{03}\,, \\ (s_-)_i = \sum_{j=1}^{3} e_j T_{ji}\,. \end{gathered} \tag{58}$$

Inserting the solutions (58) in (57), one gets

$$\|\rho_{12} - \chi_{12}\|^2 = \frac{1}{4}\big[\mathrm{Tr}K - \mathbf{e}^t K \mathbf{e}\big]\,, \tag{59}$$

where the matrix K is defined by

$$K = \mathrm{diag}(T_{11}^2, T_{22}^2, T_{30}^2 + T_{33}^2)\,. \tag{60}$$

From (19), the eigenvalues of the matrix K (60) read

$$\lambda_1 = \frac{(1-p^2)^2}{(1+p^n\cos m\pi)^2}\,, \tag{61}$$

$$\lambda_2 = \frac{(1-p^2)^2 p^{2(n-2)}}{(1+p^n\cos m\pi)^2}\,, \tag{62}$$

$$\lambda_3 = \frac{(p^2 + p^{2(n-2)})(1+p^2) + 4p^n\cos m\pi}{(1+p^n\cos m\pi)^2}\,. \tag{63}$$

It easily seen from (59) that the minimal Hilbert–Schmidt distance is obtained for the vector $\mathbf{e}$ associated with the maximal eigenvalue $\lambda_{\max}$ of the matrix K. Thus, the geometric measure of quantum discord in the state ρ_{12} is given by

$$D_{\mathrm{g}}(\rho_{12}) = \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_{\max})\,. \tag{64}$$

From the expressions (61) and (62), we have $\lambda_2 < \lambda_1$. This implies that $\lambda_{\max}$ is equal to λ_1 or λ_3. In this respect, to find the closest classical states, two situations must be considered separately. We begin with the first case where $\lambda_{\max} = \lambda_3$. The eigenvector, associated with this maximal eigenvalue, is $\mathbf{e} = (e_1 = 0, e_2 = 0, e_3 = 1)^t$. Reporting this result in (58), it is simple to check that the closest classical state (56) takes the form:

$$\chi_{12} = \frac{1}{4}\big[\sigma_0\otimes\sigma_0 + T_{30}\sigma_3\otimes\sigma_0 + T_{03}\sigma_0\otimes\sigma_3 + T_{33}\ \sigma_3\otimes\sigma_3\big]\,. \tag{65}$$

Similarly the eigenvector associated to $\lambda_{\max} = \lambda_1$ is $\mathbf{e} = (e_1 = 1, e_2 = 0, e_3 = 0)^t$ and from (58), one gets

$$\chi_{12} = \frac{1}{4}\big[\sigma_0\otimes\sigma_0 + T_{03}\sigma_0\otimes\sigma_3 + T_{11}\ \sigma_1\otimes\sigma_1\big]. \tag{66}$$

Beside the explicit derivation of closest classical states (65) and (66), another important point to be emphasized is the relation between the matrix K (60), which encodes the geometric measure quantum correlations in the state ρ_{12}, and the Bloch components of the one-qubit density matrices ρ^{ii} $(i = 1, 2)$ and ρ^{ij} $(i \neq j)$ given respectively by (15) and (16). For this end, using the relations (18), the matrix K (60) rewrites as

$$K = \mathrm{diag}(2(T_1^{01})^2, -2(T_2^{01})^2, (T_3^{00})^2 + (T_3^{11})^2)\,. \tag{67}$$

Furthermore, for one-qubit states ρ^{00}, ρ^{01}, ρ^{10} and ρ^{11}, we introduce the analogs of the matrix K (60). Hence, for the states ρ^{00} and ρ^{11} (15), we introduce the 3×3 matrices:

$$K^{kk} = (0, 0, T_3^{kk})^t(0, 0, T_3^{kk}), \quad k = 0, 1$$

and similarly, we introduce the matrices:

$$K^{kl} = (T_1^{kl}, iT_2^{kl}, 0)^t(T_1^{kl}, iT_2^{kl}, 0)\,, \quad \text{for} \quad (k, l) = (0, 1) \quad \text{or} \quad (1, 0)\,,$$

for the states ρ^{01} and ρ^{10} (16). Subsequently, one verifies

$$K = 2(K^{00} + K^{01} + K^{10} + K^{11})\,.$$

This remarkable relation holds also for the states containing three or more qubits as a consequence of the symmetry invariance of the multiqubit system under consideration. A detailed analysis of this issue is presented in what follows.

4.2. *Three-qubit states*

We now face the problem of finding the pairwise quantum discord in the three-qubit states of the form (20). This extends the results presented in the previous subsection. More especially, we analytically determine the pairwise quantum discord between the qubit one and the subsystem (23) in the state ρ_{123} (20) and we find the closest classical tripartite states. To achieve this, we write the density matrix (33) as follows:

$$\rho_{123} = \frac{1}{8}\Bigg[T_{000}\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + T_{300}\ \sigma_3 \otimes \sigma_0 \otimes \sigma_0 + \sum_{(\beta,\gamma)\neq(0,0)} T_{0\beta\gamma}\ \sigma_0 \otimes \sigma_\beta \otimes \sigma_\gamma$$
$$+ \sum_i \sum_{(\beta,\gamma)\neq(0,0)} T_{i\beta\gamma}\sigma_i \otimes \sigma_\beta \otimes \sigma_\gamma\Bigg]\,. \tag{68}$$

The classical states (i.e., states presenting zero discord between the qubit one and the subsystem (23) are of the form

$$\chi_{1|23} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^{23} + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^{23} \tag{69}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit one. The density matrices ρ_i^{23} $(i = 1, 2)$ corresponding to the [subsystem (23)] write as

$$\rho_i^{23} = \frac{1}{4}\Bigg[\sum_{\alpha,\beta} \mathrm{Tr}(\rho_i^{23}\sigma_\alpha \otimes \sigma_\beta)\sigma_\alpha \otimes \sigma_\beta\Bigg]\,.$$

The Fano–Bloch form of the tripartite classical state (69) is given by

$$\chi_{1|23} = \frac{1}{8}\Bigg[\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} te_i\sigma_i \otimes \sigma_0 \otimes \sigma_0 + \sum_{(\alpha,\beta)\neq(0,0)} (s_+)_{\alpha,\beta}\sigma_0 \otimes \sigma_\alpha \otimes \sigma_\beta + \sum_{i=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)} e_i(s_-)_{\alpha,\beta}\sigma_i \otimes \sigma_\alpha \otimes \sigma_\beta\Bigg], \tag{70}$$

where

$$t = p_1 - p_2\,, \quad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle\,, \quad (s_\pm)_{\alpha,\beta} = \mathrm{Tr}\big((p_1\rho_1^{23} \pm p_2\rho_2^{23})\sigma_\alpha \otimes \sigma_\beta\big).$$

The Hilbert–Schmidt distance between the three-qubit state ρ_{123} (68) and a classical state (70) is

$$\|\rho_{1|23} - \chi_{1|23}\|^2 = \frac{1}{8}\Bigg[(t^2 - 2te_3T_{300} + T_{300}^2) + \sum_{(\alpha,\beta)\neq(0,0)} (T_{0\alpha\beta} - (s_+)_{\alpha,\beta})^2 + \sum_{i=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)} (T_{i\alpha\beta} - e_i(s_-)_{\alpha,\beta})^2\Bigg]. \tag{71}$$

To derive the closest classical state as measured by Hilbert–Schmidt, an optimization with respect to the parameters t, e_i $(i = 1,2,3)$ and $(s_\pm)_{\alpha,\beta}$ is performed. Thus, the minimal distance is attainable by setting zero the partial derivatives of the Hilbert–Schmidt distance (71) with respect to t and $(s_\pm)_{\alpha,\beta}$. This gives

$$t = e_3T_{300}\,, \quad (s_+)_{\alpha,\beta} = T_{0\alpha\beta}, \quad (s_-)_{\alpha,\beta} = \sum_{i=1}^{3} e_iT_{i\alpha\beta}. \tag{72}$$

Reporting the results (72) in (71), one obtains

$$\|\rho_{1|23} - \chi_{1|23}\|^2 = \frac{1}{8}\Bigg[T_{300}^2 - e_3^2T_{300}^2 + \sum_{i=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)} T_{i\alpha\beta}^2 - \sum_{i,j=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)} e_ie_jT_{i\alpha\beta}T_{j\alpha\beta}\Bigg], \tag{73}$$

to be optimized with respect to the three components of the unit vector $\mathbf{e} = (e_1, e_2, e_3)$. Equation (73) can reexpressed as

$$\|\rho_{1|23} - \chi_{1|23}\|^2 = \frac{1}{8}\big[\|x\|^2 + \|T\|^2 - \mathbf{e}(xx^t + TT^t)\mathbf{e}^t\big]\,, \tag{74}$$

in terms of the 1×3 matrix defined by

$$x^t := (0, 0, T_{300}) \tag{75}$$

and the 3×15 matrix given by

$$T := (T_{i\alpha\beta}) \quad \text{with} \quad (\alpha, \beta) \neq (0, 0)\,. \tag{76}$$

Setting

$$K = xx^t + TT^t \tag{77}$$

and reporting (75) and (76) in (77), one obtains after some tedious calculations:

$$K = \mathrm{diag}(k_1, k_2, k_3)\,, \tag{78}$$

where k_1, k_2 and k_3 are given by

$$\begin{gathered} k_1 = \sum_{i=1,2}\sum_{j=0,3} T_{1ij}^2 + T_{1ji}^2\,, \quad k_2 = \sum_{i=1,2}\sum_{j=0,3} T_{2ij}^2 + T_{2ji}^2\,, \\ k_3 = \sum_{i=0,3}\sum_{j=0,3} T_{3ij}^2 + \sum_{i=1,2}\sum_{j=1,2} T_{3ij}^2\,. \end{gathered} \tag{79}$$

Using the relations (34) and (35), the eigenvalues of the matrix K can be reexpressed in terms of the bipartite correlations elements $T_{\alpha\beta}$ associated with the qubit density matrices ρ^{01}, ρ^{01}, ρ^{10} and ρ^{11} (*cf.* (28) and (30)). Therefore, one has

$$k_1 = 2[(T_{11}^{00})^2 + (T_{11}^{11})^2] + 4|T_{10}^{01}|^2 + 4|T_{13}^{01}|^2, \tag{80}$$

$$k_2 = 2[(T_{22}^{00})^2 + (T_{22}^{11})^2] + 4|T_{20}^{01}|^2 + 4|T_{23}^{01}|^2, \tag{81}$$

$$k_3 = 2[(T_{30}^{00})^2 + (T_{30}^{11})^2] + 2[(T_{33}^{00})^2 + (T_{33}^{11})^2] + 4|T_{31}^{01}|^2 + 4|T_{32}^{01}|^2. \tag{82}$$

Finally, using (29) and (31), we obtain

$$k_1 = 2\frac{(1-p^2)^2(1+p^2)}{(1+p^n\cos m\pi)^2}\,, \tag{83}$$

$$k_2 = 2\frac{(1-p^2)^2(1+p^2)p^{2(n-3)}}{(1+p^n\cos m\pi)^2}\,, \tag{84}$$

$$k_3 = 2\frac{(p^2+p^{2(n-3)})(1+p^4)+4p^n\cos m\pi}{(1+p^n\cos m\pi)^2}\,, \tag{85}$$

The minimal value of the Hilbert–Schmidt distance (74) is reached when $\mathbf{e}$ is the eigenvector associated to the largest eigenvalue of the matrix defined by (77). We denote by $k_{\max}$ the largest eigenvalue among k_1, k_2 and k_3. Since $k_1 \geq k_2$, $k_{\max}$ is k_2 or k_3 depending on the number of qubits n and the overlap p. Notice that the sum of the eigenvalues k_1, k_2 and k_3 of the matrix K is exactly the sum of the Hilbert–Schmidt norm of the matrices x (75) and T (76) (i.e. $k_1 + k_2 + k_3 = \|x\|^2 + \|T\|^2$). It follows that the minimal Hilbert–Schmidt distance (74) writes as,

$$D_{\mathrm{g}}(\rho_{1|23}) = \frac{1}{8}(k_1 + k_2 + k_3 - k_{\max}) \tag{86}$$

and gives the geometric measure of the pairwise quantum discord in the state ρ_{123} partitioned in the subsystems (1) and (23). When the matrix elements of the density matrix ρ_{123} (32) are such that $k_{\max} = k_1$, one can simply verify that the closest classical state is given by

$$\chi^{(1)}_{1|23} = \frac{1}{8}\left[\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sum_{(\alpha,\beta)\neq(0,0)} T_{0\alpha\beta}\sigma_0 \otimes \sigma_\alpha \otimes \sigma_\beta + \sum_{(\alpha,\beta)\neq(0,0)} T_{1\alpha\beta}\sigma_1 \otimes \sigma_\alpha \otimes \sigma_\beta\right]. \tag{87}$$

Conversely, in the situation where $k_{\max} = k_3$, one finds

$$\chi^{(3)}_{1|23} = \frac{1}{8}\left[\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + T_{300}\sigma_3 \otimes \sigma_0 \otimes \sigma_0 + \sum_{(\alpha,\beta)\neq(0,0)} T_{0\alpha\beta}\sigma_0 \otimes \sigma_\alpha \otimes \sigma_\beta \right.$$
$$\left. + \sum_{(\alpha,\beta)\neq(0,0)} T_{3\alpha\beta}\sigma_3 \otimes \sigma_\alpha \otimes \sigma_\beta\right]. \tag{88}$$

4.3. *k-qubit states*

Now we come to the generalization of the previous analysis. In this order, we shall determine the explicit expression of the geometric discord in the k-qubit state (36) when a bipartite splitting of type $1|23\cdots k$ is considered. We also derive the closest classical state to the state (36). We first expand the density matrix $\rho_{12\cdots k}$ (42) as

$$\rho_{12\cdots k} = \frac{1}{2^k}\left[T_{00\cdots 0}\sigma_0 \otimes \sigma_0 \cdots \otimes \sigma_0 + T_{30\cdots 0}\sigma_3 \otimes \sigma_0 \otimes \cdots \sigma_0\right.$$
$$+ \sum_{(\alpha_2,\ldots\alpha_k)\neq(0,\ldots,0)} T_{0\alpha_2\cdots\alpha_k}\sigma_0 \otimes \sigma_{\alpha_2} \otimes \cdots \sigma_{\alpha_k}$$
$$\left. + \sum_i \sum_{(\alpha_2,\ldots\alpha_k)\neq(0,\ldots,0)} T_{i\alpha_2\cdots\alpha_k}\sigma_i \otimes \sigma_{\alpha_2} \otimes \cdots \sigma_{\alpha_k}\right], \tag{89}$$

in terms of the nonvanishing correlations coefficients. Any k-qubit state having zero discord is necessarily of the form:

$$\chi_{1|23\cdots k} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^{23\cdots k} + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^{23\cdots k}, \tag{90}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit one. The density matrices $\rho_i^{23\cdots k}$ $(i = 1, 2)$, corresponding to the subsystem $(23\cdots k)$, that contains $(k-1)$ qubits, write as

$$\rho_i^{23\cdots k} = \frac{1}{2^{k-1}}\left[\sum_{\alpha_2,\ldots\alpha_k} \mathrm{Tr}(\rho_i^{23\cdots k}\sigma_{\alpha_2} \otimes \cdots \sigma_{\alpha_k})\sigma_{\alpha_2} \otimes \cdots \sigma_{\alpha_k}\right].$$

To examine the pairwise quantum correlations in the states (36), the appropriate form for multipartite classical state (90) is

$$\chi_{1|23\cdots k} = \frac{1}{2^k}\Bigg[\sigma_0\otimes\sigma_0\cdots\otimes\sigma_0 + \sum_{i=1}^{3} te_i\sigma_i\otimes\sigma_0\cdots\otimes\sigma_0 + \sum_{(\alpha_2,\ldots\alpha_k)\neq(0,\ldots,0)} (s_+)_{\alpha_2,\ldots,\alpha_k}\sigma_0\otimes\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k} + \sum_{i=1}^{3}\sum_{(\alpha_2,\ldots,\alpha_k)\neq(0,\ldots,0)} e_i(s_-)_{\alpha_2,\ldots,\alpha_k}\sigma_i\otimes\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k}\Bigg], \tag{91}$$

where

$$t = p_1 - p_2\,, \quad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle\,,$$
$$(s_\pm)_{\alpha_2,\ldots,\alpha_k} = \mathrm{Tr}\big((p_1\rho_1^{23\cdots k} \pm p_2\rho_2^{23\cdots k})\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k}\big)\,.$$

Hence, the Hilbert–Schmidt distance between the state $\rho_{123\cdots k}$ (89) and a classical state of the form (91) is given by the following expression:

$$\|\rho_{1|23\cdots k} - \chi_{1|23\cdots k}\|^2 = \frac{1}{2^k}\Bigg[(t^2 - 2te_3T_{30\cdots 0} + T_{30\cdots 0}^2) + \sum_{(\alpha_2,\ldots,\alpha_k)\neq(0,\ldots,0)} (T_{0\alpha_2\cdots\alpha_k} - (s_+)_{\alpha_2,\ldots,\alpha_k})^2 + \sum_{i=1}^{3}\sum_{(\alpha_2,\ldots,\alpha_k)\neq(0,\ldots,0)} (T_{i\alpha_2\cdots\alpha_k} - e_i(s_-)_{\alpha_2,\ldots,\alpha_k})^2\Bigg], \tag{92}$$

which must be optimized with respect to the parameters t, e_i ($i = 1,2,3$) and $(s_\pm)_{\alpha_2,\ldots,\alpha_k}$ to find the closest classical states. In this sense, we start by setting the partial derivatives of (92), with respect to the parameters t and $(s_\pm)_{\alpha_2,\ldots,\alpha_k}$, equal to zero. Thus, we get

$$t = e_3T_{30\cdots 0}\,, \quad (s_+)_{\alpha_2,\ldots,\alpha_k} = T_{0\alpha_2\cdots\alpha_k}\,, \quad (s_-)_{\alpha_2,\ldots,\alpha_k} = \sum_{i=1}^{3} e_iT_{i\alpha_2\cdots\alpha_k}\,. \tag{93}$$

Reporting the conditions (93) in the expression (92), one obtains

$$\|\rho_{1|23\cdots k} - \chi_{1|23\cdots k}\|^2 = \frac{1}{2^k}\Bigg[T_{30\cdots 0}^2 - e_3^2T_{30\cdots 0}^2 + \sum_{i=1}^{3}\sum_{(\alpha_2,\ldots,\alpha_k)\neq(0,0)} T_{i\alpha_2\cdots\alpha_k}^2 - \sum_{i,j=1}^{3}\sum_{(\alpha_2,\ldots,\alpha_k)\neq(0,\ldots,0)} e_ie_jT_{i\alpha_2\cdots\alpha_k}T_{j\alpha_2\cdots\alpha_k}\Bigg]\,, \tag{94}$$

that has to be optimized with respect to the three components of the unit vector $\mathbf{e} = (e_1, e_2, e_3)$ in order to get the minimal Hilbert–Schmidt distance. After some algebra, the distance (94) takes the following compact form:

$$\|\rho_{1|23\cdots k} - \chi_{1|23\cdots k}\|^2 = \frac{1}{2^k}\left[\|x\|^2 + \|T\|^2 - \mathbf{e}(xx^t + TT^t)\mathbf{e}^t\right], \tag{95}$$

in terms of the 1×3 matrix defined by

$$x^t = (0, 0, T_{30\cdots 0}) \tag{96}$$

and the $3 \times (4^{k-1} - 1)$ matrix given by

$$T = (T_{i\alpha_2\cdots\alpha_k}) \quad \text{with} \quad (\alpha_2, \ldots, \alpha_k) \neq (0, \ldots, 0), \tag{97}$$

which are the extended versions of the matrices (75) and (76) introduced for $k = 3$. Similarly to the particular cases $k = 2, 3$ and from Eq. (95), it is easily seen that the pairwise quantum correlation is completely characterized by the eigenvalues of the matrix:

$$K = xx^t + TT^t. \tag{98}$$

It is clear that the computation of these eigenvalues for an arbitrary multiqubit state constitutes a very complex task. However, this complexity is considerably reduced for the states $\rho_{12\cdots k}$ by exploiting their parity symmetry (i.e., commutes with $\sigma_3 \otimes \sigma_3 \cdots \sigma_3$). This implies that the matrix T (97) writes formally as

$$T = \sum_{\alpha_2,\ldots,\alpha_k} (T_{1\alpha_2\cdots\alpha_k}, \quad T_{2\alpha_2\cdots\alpha_k}, 0)^t + \sum_{\alpha_2,\ldots,\alpha_k} (0, 0, T_{3\alpha_2\cdots\alpha_k})^t.$$

This form is more appropriate to show that the product TT^t is diagonal. The qubits forming the system described by the state $\rho_{12\cdots k}$ are identical and invariant under exchange symmetry. Consequently, since the elements the density matrix $\rho_{12\cdots k}$ are real values and in view of the recurrence relations (43), the off-diagonal entries of the matrix TT^t vanish. This result has been discussed already for $k = 2$, $k = 3$ and will be explicitly proved hereafter for $k = 4$. It follows that the matrix K (98) is diagonal:

$$K = \mathrm{diag}(k_1, k_2, k_3),$$

where

$$k_1 = \sum_{\alpha_2,\ldots,\alpha_k} T^2_{1\alpha_2\cdots\alpha_k},$$

$$k_2 = \sum_{\alpha_2,\ldots,\alpha_k} T^2_{2\alpha_2\cdots\alpha_k},$$

$$k_3 = T^2_{30\cdots 0} + \sum_{\alpha_2,\ldots,\alpha_k \neq 0} T^2_{3\alpha_2\cdots\alpha_k}.$$

To exemplify this procedure, we consider the situation where $k = 4$. In this case, the 3×63 matrix elements of T defined by (97) can be explicitly derived using

Eq. (52). A straightforward but lengthy computation shows that the 3×3 matrix K is diagonal and the corresponding eigenvalues are

$$k_1 = \sum_{k=0,3} \sum_{i=1,2} \sum_{j=0,3} T^2_{1kji} + \sum_{k=1,2} \sum_{j=1,2} \sum_{i=1,2} T^2_{1kji} + \sum_{k=0,3} \sum_{j=1,2} \sum_{i=0,3} T^2_{1kji} + \sum_{k=1,2} \sum_{j=0,3} \sum_{i=0,3} T^2_{1kji}, \tag{99}$$

$$k_2 = \sum_{k=0,3} \sum_{i=1,2} \sum_{j=0,3} T^2_{2kji} + \sum_{k=1,2} \sum_{j=1,2} \sum_{i=1,2} T^2_{2kji} + \sum_{k=0,3} \sum_{j=1,2} \sum_{i=0,3} T^2_{2kji} + \sum_{k=1,2} \sum_{j=0,3} \sum_{i=0,3} T^2_{2kji}, \tag{100}$$

$$k_3 = \sum_{k=0,3} \sum_{i=0,3} \sum_{j=0,3} T^2_{3kji} + \sum_{k=1,2} \sum_{j=1,2} \sum_{i=0,3} T^2_{3kji} + \sum_{k=0,3} \sum_{j=1,2} \sum_{i=1,2} T^2_{3kji} + \sum_{k=1,2} \sum_{j=0,3} \sum_{i=1,2} T^2_{3kji}. \tag{101}$$

The expressions (99), (100) and (101) can be simplified further. Indeed, from the relations (43), which reproduce the expressions (52) for $k = 4$, one obtains

$$k_1 = 2\left[\sum_{k=0,3} \sum_{j=1,2} ((T^{00}_{1kj})^2 + (T^{11}_{1jk})^2) + \sum_{k=1,2} \sum_{j=0,3} ((T^{00}_{1kj})^2 + (T^{11}_{1jk})^2)\right] + 4\left[\sum_{k=0,3} \sum_{j=0,3} T^{01}_{1kj} T^{10}_{1kj} + \sum_{k=1,2} \sum_{j=1,2} T^{01}_{1kj} T^{10}_{1kj}\right], \tag{102}$$

$$k_2 = 2\left[\sum_{k=0,3} \sum_{j=1,2} ((T^{00}_{2kj})^2 + (T^{11}_{2jk})^2) + \sum_{k=1,2} \sum_{j=0,3} ((T^{00}_{2kj})^2 + (T^{11}_{2jk})^2)\right] + 4\left[\sum_{k=0,3} \sum_{j=0,3} T^{01}_{2kj} T^{10}_{2kj} + \sum_{k=1,2} \sum_{j=1,2} T^{01}_{2kj} T^{20}_{1kj}\right], \tag{103}$$

$$k_3 = 2\left[\sum_{k=0,3} \sum_{j=0,3} ((T^{00}_{3kj})^2 + (T^{11}_{3jk})^2) + \sum_{k=1,2} \sum_{j=1,2} ((T^{00}_{3kj})^2 + (T^{11}_{3jk})^2)\right] + 4\left[\sum_{k=0,3} \sum_{j=1,2} T^{01}_{3kj} T^{10}_{3kj} + \sum_{k=1,2} \sum_{j=0,3} T^{01}_{3kj} T^{10}_{3kj}\right], \tag{104}$$

in terms of the three-qubit correlation elements $T^{kl}_{\alpha\beta\gamma}$ associated with the density matrices ρ^{kl}_{123} (53). The tripartite correlations coefficients $T^{kl}_{\alpha\beta\gamma}$ are evaluated using the recurrence relations of type (34) and (35) (modulo some obvious substitution) as expansion of bipartite correlations associated with two-qubit subsystems. Subsequently, one finds

$$k_1 = 16\mathcal{N}^4(1-p^2)(1-p^6)\,, \tag{105}$$

$$k_2 = 16\mathcal{N}^4(1-p^2)(1-p^6)p^{2(n-4)}\,, \tag{106}$$

$$k_3 = 16\mathcal{N}^4\left[(1+p^6)(p^2+p^{2(n-4)}) + 4p^n\cos m\pi\right]. \tag{107}$$

Clearly, the derivation of pairwise quantum discord in k-qubit mixed states between one qubit and the other $(k-1)$ qubits, viewed as a single subsystem, requires tedious analytical manipulation. However, it must be noticed that the parity containing $(k-1)$ qubits can be mapped onto two logical qubits. This encoding scheme was recently considered in Refs. 36, 45 and 46 to examine the pairwise quantum correlations in multiqubit systems. In this spirit, we shall compare in the following section the geometric measure of quantum discord obtained in each picture.

5. Pairwise Encoding

Different suitable splitting scenarios are possible in investigating quantum correlations in a n-qubit system. In the previous sections, we essentially focused on the quantum correlation in k-qubit states ($k = 0, 1, \ldots, n-1$) extracted by a trace procedure from the whole system, by splitting the system of k qubits into a single qubit and a cluster of $(k-1)$ qubits. In this section, we shall consider the scenario where the information contained in the cluster of $(k-1)$ particles is encoded into two logical qubits $\{|0\rangle_{23\cdots k}, |1\rangle_{23\cdots k}\}$ defined by

$$|\eta,\eta,\ldots,\eta\rangle \equiv b_+|0\rangle_{23\cdots k} + b_-|1\rangle_{23\cdots k}\,, \quad |-\eta,-\eta,\ldots,-\eta\rangle \equiv b_+|0\rangle_{23\cdots k} - b_-|1\rangle_{23\cdots k}\,, \tag{108}$$

where

$$b_\pm = \sqrt{\frac{1\pm p^{k-1}}{2}}\,.$$

In this encoding scheme, the density matrix $\rho_{1|23\cdots k} \equiv \rho_{1(23\cdots k)}$ (36) rewrites, in the basis $\{|0\rangle\otimes|0\rangle_{23\cdots k}, |0\rangle\otimes|1\rangle_{23\cdots k}, |1\rangle\otimes|0\rangle_{23\cdots k}, |1\rangle\otimes|1\rangle_{23\cdots k}\}$, as

$$\rho_{1(23\cdots k)} = 2\mathcal{N}^2\begin{pmatrix} a_+^2b_+^2(1+q_k\cos m\pi) & 0 & 0 & a_+a_-b_+b_-(1+q_k\cos m\pi) \\ 0 & a_+^2b_-^2(1-q_k\cos m\pi) & a_+a_-b_+b_-(1-q_k\cos m\pi) & 0 \\ 0 & a_+a_-b_+b_-(1-q_k\cos m\pi) & a_-^2b_+^2(1-q_k\cos m\pi) & 0 \\ a_+a_-b_+b_-(1+q_k\cos m\pi) & 0 & 0 & a_-^2b_-^2(1+q_k\cos m\pi) \end{pmatrix}, \tag{109}$$

or equivalently, in the Fano–Bloch representation, as

$$\rho_{1(23\cdots k)} = \frac{1}{4}\sum_{\alpha\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta , \tag{110}$$

where the nonvanishing matrix elements $R_{\alpha\beta}$ $(\alpha,\beta = 0,1,2,3)$ are given by

$$R_{00} = 1\,, \quad R_{11} = 2\mathcal{N}^2\sqrt{(1-p^2)(1-p^{2(k-1)})}\,,$$

$$R_{22} = -2\mathcal{N}^2\sqrt{(1-p^2)(1-p^{2(k-1)})}\; p^{n-k}\cos m\pi\,,$$

$$R_{33} = 2\mathcal{N}^2(p^k + p^{n-k}\cos m\pi)\,, \quad R_{03} = 2\mathcal{N}^2(p^{k-1} + p^{n-k+1}\cos m\pi)\,,$$

$$R_{30} = 2\mathcal{N}^2(p + p^{n-1}\cos m\pi)\,.$$

Following the standard procedure to derive the geometric discord for a two-qubit system, it is simple to check that

$$D_{\rm g}(\rho_{1(23\cdots k)}) = \frac{1}{4}\min\{l_1 + l_2, l_1 + l_3, l_2 + l_3\}\,. \tag{111}$$

where

$$l_1 = R_{11}^2, \quad l_2 = R_{22}^2, \quad l_3 = R_{30}^2 + R_{33}^2.$$

Explicitly, the quantities λ_1, λ_2 and λ_3 are given by

$$l_1 = 4\mathcal{N}^4(1-p^2)(1-p^{2(k-1)})\,, \tag{112}$$

$$l_2 = 4\mathcal{N}^4(1-p^2)(1-p^{2(k-1)})p^{2(n-k)}\,, \tag{113}$$

$$l_3 = 4\mathcal{N}^4\big[(1+p^{2(k-1)})(p^2 + p^{2(n-k)}) + 4p^n\cos m\pi\big]\,. \tag{114}$$

It is remarkable that for $k = 2$, $k = 3$ and $k = 4$, one recovers the results (61)–(63), (83)–(85) and (105)–(107) respectively (up to the overall multiplicative factor 2^{k-2}). Indeed, we have

$$D_{\rm g}(\rho_{1(23\cdots k)}) = \frac{1}{2^{k-2}} D_{\rm g}(\rho_{1|23\cdots k})\,.$$

This shows that encoding $(k-1)$ qubits in two logical qubits constitutes an alternative and efficient way to compute easily the geometric measure of quantum discord.

6. Concluding Remarks

In this paper, we developed a general algorithm to evaluate the pairwise geometric discord in a mixed state $\rho_{123\cdots k}$ comprised of k qubits. This provides a closed analytical expressions for the geometric quantum discord based on Hilbert–Schmidt distance. We especially considered multiqubit states possessing parity invariance and exchange symmetry. A detailed analysis is performed for reduced density matrices

$\rho_{123\cdots k}$ obtained by a trace procedure from a balanced superpositions of symmetric n-qubit states. Two splitting schemes were discussed. In the first one, where the reduced density is denoted by $\rho_{1|23\cdots k}$, a recursive algorithm is proposed to determine explicitly the pairwise geometric discord between the first qubit and the remaining $(k-1)$ qubits. The parity and exchange symmetries simplify considerably the determination of the geometric measure of quantum discord. The recursive approach offers a very useful prescription to determine geometric quantum discord in terms of two-qubit correlation matrices. This constitutes the key ingredient in deriving the geometric discord. Another important issue we examined in this work concerns the explicit derivation of classical (zero discord) states. We have also shown that there exists an alternative scheme offering a simple procedure to compute the geometric discord. This uses a bipartition scheme according to which the system grouping the $(k-1)$ qubits in the state $\rho_{123\cdots k}$ is mapped into a set of two logical qubits. Remarkably the two schemes lead to the same result for the Hilbert–Schmidt measure of pairwise geometric discord.

We believe that the results obtained in this work can be extended to other classes of multiqubit states. We also notice that they can be exploited in evaluating multipartite geometric quantum discord in the spirit of the results recently obtained in Ref. 44. Finally, another interesting application of the results obtained here, that deserve a special attention, concerns the distribution of geometric quantum discord between the different components of a multiqubit system.

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