Fisher information and LQU for two-qubit Bell states under quantum decoherence channel

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Abstract

Il s'agit d'etudier l'evolution de l'information de Fisher et la LQU (local quantum uncertainty) pour des etats X a deux qubits quand ils evoulent dans un canal decoherent. Bien sur, il s'agit de voir quelle est l'inflence de l'environnement en metrologie.

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1 Introduction

A class of two-qubit states called X-states are increasingly being used to discuss entanglement and other quantum correlations in the field of quantum information. Maximally entangled Bell states and "Werner" states are subsets of them. Apart from being so named because their density matrix looks like the letter X, there is not as yet any characterization of them. The $su(2) \times su(2) \times u(1)$ subalgebra of the full $su(4)$ algebra of two qubits is pointed out as the underlying invariance of this class of states. X-states are a seven-parameter family associated with this subalgebra of seven operators. This recognition provides a route to preparing such states and also a convenient algebraic procedure for analytically calculating their properties. At the same time, it points to other groups of seven-parameter states that, while not at first sight appearing similar, are also invariant under the same subalgebra. And it opens the way to analyzing invariant states of other subalgebras in bipartite systems.

Increasingly in the field of quantum information, aspects of entanglement [46], and of other quantum correlations such as, for instance, "quantum discord" [47], between two qubits have been described for a class of pure and mixed states that have come to be called "X-states" [48]. Although their use goes back further [49], they were so named in [48] because of the visual appearance of the density matrix, that it looks like the letter in the alphabet:

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{1}$$

Non-zero entries occur only along the diagonal and anti-diagonal. Many calculations of entanglement and other properties [49, 50], and their evolution under unitary or dissipative processes [51], can be carried out analytically for such states which make them appealing objects for study. Many other states of interest, such as the maximally entangled Bell states [46] and "Werner" states [52], are a sub-class of X-states, lending further importance to their study.

Yet, no firmer definition has been given of what makes a pure or mixed system an X-state. This Letter provides such a definition in terms of their invariance properties, that a particular symmetry group or algebra underlies them. Such an identification of an underlying symmetry helps to explain the analytical results while at the same time providing a well defined procedure for their preparation. Recognizing the symmetry also makes computations involving such states, such as unitary operations on them or evaluating concurrence or other measures of entanglement, straightforward and easily tractable. And, finally, the symmetry also opens the way for constructing other density matrices which may not visually appear as X but are nevertheless similar, states of a different rendering of the same algebraic symmetry. Since they differ in entanglement and separability considerations, they may prove useful for study.

2 The subalgebra of X-states

Positivity and other standard requirements of any density matrix make the X-states shown in Eq. (1) a seven-parameter family. The diagonal elements of the density matrix are real so that, along with the trace being fixed at 1, three real parameters describe those diagonal entries. Hermiticity to guarantee real eigenvalues reduces the off-diagonal entries to two complex (say ρ_{14} and ρ_{23}, with ρ_{41} and ρ_{32} their respective complex conjugates) or four real parameters for the total of seven real parameters.

The full two qubit system has the symmetry of the SU(4) group and its algebra $su(4)$. Fifteen operators, most conveniently rendered as fifteen linearly independent 4×4 matrices or as Pauli spinors/matrices of the two spins, together with the unit matrix, provide a complete description of the general system. There are, however, several subalgebras of $su(4)$. A series of recent papers have provided a geometrical description of their states and operators [53, 54, 55, 56]. In particular, one subalgebra, $su(2) \times su(2) \times u(1)$, of seven operators or matrices occurs in many physical systems in quantum optics and quantum information [53, 54]. This Letter presents them as the invariance set of the X-states.

Inspection of the explicit 4×4 matrices in a standard basis for two spins, $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$, is instructive [53, 54, 57, 58] and points immediately to sets of seven of them with the same structure of eight zeroes in the same positions as in Eq. (1). That is, these are operators that do not mix the 1-4 and 2-3 subspaces of the density matrix. Combined with the observation that such a set of seven matrices closes under multiplication, it is immediate that they will carry X-states into each other, that they preserve the X structure. For this purpose, both the Lie algebra aspect that the seven operators close under commutation and their Clifford algebraic structure that they close under multiplication are important. Indeed, explicit rendering of the fifteen operators in terms of two Pauli spinors called $\vec{\sigma}$ and $\vec{\tau}$, together with the familiar multiplication rule $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, i, j, k = 1-3$, where ϵ_{ijk} is the completely antisymmetric symbol and repeated indices are summed, is very useful for operations with them.

There are many such sets of seven operators/matrices constituting the $su(2) \times su(2) \times u(1)$ subalgebra [53, 54, 56]. In each of them, one operator, the $u(1)$ element, commutes with all six of the others which themselves can be further subdivided as shown in [53] into two sets of "pseudospins", two sets of three which obey commutation relations of angular momentum within each set while all three of one set commute with all three of the other. Any one of the fifteen operators can serve as the commuting element because, as shown in a table in [54], each row has six zeroes so that each identifies such a $su(2) \times su(2) \times u(1)$ set. There are, therefore, fifteen non-equivalent such subalgebras.

We will designate such a set by $\{X_i\}, i = 1, 2, \ldots, 7$, with X_1 the commuting element. One such is $(X_1 = \sigma_z\tau_z, X_2 = \sigma_y\tau_x, X_3 = \tau_z, X_4 = -\sigma_y\tau_y, X_5 = \sigma_x\tau_y, X_6 = \sigma_z, X_7 = \sigma_x\tau_x)$. This is the same set that occurs in the CNOT quantum logic gate constructed out of two Josephson junctions and was extensively studied in that context [53]. It was also pointed out that it occurs in nuclear magnetic resonance when each spin is in an external magnetic field in the z-direction while being coupled to each other through scalar coupling $\vec{\sigma} \cdot \vec{\tau}$ and “cross-coherences” $\sigma_x \tau_y$ and $\sigma_y \tau_x$. But a different choice for the commuting element X_1 gives another such subalgebra, and we will return to this in section IV. Each X_i is traceless, Hermitian, and unitary, and its square is unity so that the eigenvalues are $(\pm 1, \pm 1)$.

With any such set, $\{X_i\}$, the density matrix that remains invariant under their operations can be rendered as a linear superposition of them,

$$\rho = (I + \Sigma_i g_i X_i)/4, \tag{2}$$

in analogy to that for a single spin, $(I + \Sigma_i g_i \sigma_i)/2$. The seven real coefficients g_i in Eq. (2) parametrize X-states and are equivalent to the seven parameters in the density matrix in Eq. (1):

$$\begin{aligned}
g_1 &= (\rho_{11} + \rho_{44}) - (\rho_{22} + \rho_{33}), \\
g_2 &= 2i(\rho_{14} - \rho_{41} + \rho_{32} - \rho_{23}), \\
g_3 &= (\rho_{11} - \rho_{44}) - (\rho_{22} - \rho_{33}), \\
g_4 &= 2(\rho_{14} + \rho_{41} - \rho_{32} - \rho_{23}), \\
g_5 &= 2i(\rho_{14} - \rho_{41} - \rho_{32} + \rho_{23}), \\
g_6 &= (\rho_{11} - \rho_{44}) + (\rho_{22} - \rho_{33}), \\
g_7 &= 2(\rho_{14} + \rho_{41} + \rho_{32} + \rho_{23}).
\end{aligned} \tag{3}$$

The algebra of the seven $\{X_i\}$ is most conveniently captured by Fig. 1 as has recently been pointed out [56]. This figure occurs in projective geometry as the “Fano Plane” [59] and also is used to represent the multiplication table for octonions [60]. Arranging the seven operators at the vertices, mid-points of sides and in-center of an equilateral triangle, the seven lines shown (including the inscribed circle) each carry three points, providing the multiplication rule for those $\{X_i\}$. The notation of arrows is also borrowed from octonions except that unlike them which have all seven lines arrowed, the three internal verticals are not in Fig. 1. On those lines, all three operators mutually commute, so that the product of two gives the third regardless of order. On the four arrowed lines, the operators mutually anticommute so that the product of two gives $(\pm i)$ times the third, with plus (minus) signs along (against) the sense of the arrow. For this purpose, each line is regarded as a closed loop with a continuously circulating arrow. The central element commutes with all six of the others. For each of those, there is one “conjugate” element with which it commutes and four with which it anticommutes. All of this can be read off by merely glancing at Fig. 1 and will provide simple rules for their manipulation in the next section.

3 Fisher information for a non-full rank density matrix

4 Quantum decoherence channels

A quantum channel can be described in the Kraus representation

$$\mathcal{E}(\rho) = \sum_{\mu} K_{\mu}\rho K_{\mu}^{\dagger}, \tag{4}$$

where K_{μ} are Kraus operators satisfying $\sum_{\mu} K_{\mu}^{\dagger}K_{\mu} = I$. As we discussed in the previous section, to obtain the GMQD, we need to know the expectation values of the Pauli matrices of the two qubits for the state $\mathcal{E}(\rho)$. So we turn to the Heisenberg picture to describe quantum channels via the map [40]

$$\mathcal{E}^{\dagger}(A) = \sum_{\mu} K_{\mu}^{\dagger}AK_{\mu} \tag{5}$$

with A an arbitrary observable. Then the expectation value of A can be obtained through $\langle A\rangle = \mathrm{Tr}\left[A\mathcal{E}(\rho)\right] = \mathrm{Tr}\left[\mathcal{E}^{\dagger}(A)\rho\right]$. Because an arbitrary Hermitian operator on $\mathbb{C}^2$ can be expressed by $A = \sum_{i=0}^{3} r_i\sigma_i$ with $r_i \in \mathbb{R}$, then a quantum channel for a qubit can be characterized by the transmission matrix M defined through

$$\mathcal{E}^{\dagger}(\sigma_i) = \sum_{j} M_{ij}\sigma_j \quad \text{or} \quad M_{ij} = \frac{1}{2}\mathrm{Tr}\left[\mathcal{E}^{\dagger}(\sigma_i)\sigma_j\right]. \tag{6}$$

Since $\mathrm{Tr}[\mathcal{E}^{\dagger}(\sigma_i)\rho] = \sum_j M_{ij}\mathrm{Tr}[\sigma_j\rho]$, M_{ij} actually describes the transformation of the polarized vector $P_i \equiv \mathrm{Tr}[\sigma_j\rho]$.

Now we consider the case of two qubits under local decoherence channels, i.e., $\rho = [\mathcal{E}_A \otimes \mathcal{E}_B](\rho_0)$. To obtain the GMQD of the output state ρ through the channel, we need to get the expectation matrix $\mathcal{R}$. With the Heisenberg picture, we have

$$\mathcal{R}_{ij} = \mathrm{Tr}(\mathcal{E}_A^{\dagger}(\sigma_i) \otimes \mathcal{E}_B^{\dagger}(\sigma_j)\rho_0) = (M_A\mathcal{R}_0M_B^T)_{ij}, \tag{7}$$

where $\mathcal{R}_0$ is the expectation matrix under ρ_0, i.e., $(\mathcal{R}_0)_{ij} = \mathrm{Tr}(\sigma_i \otimes \sigma_j\rho_0)$, and $M_{A(B)}$ is the transformation matrix characterizing the quantum channel $\mathcal{E}_{A(B)}$. So we obtain $\mathcal{R} = M_A\mathcal{R}_0M_B^T$.

For simplicity, we assume $\mathcal{E}^A$ and $\mathcal{E}^B$ be identical, hereafter. Next, we consider three typical kinds of decoherence channels: the amplitude damping channel (ADC), the phase damping channel (PDC), and the depolarizing channel (DPC). They are described by the set of Kraus operators respectively [42, 41]:

$$\begin{aligned}
K^{\mathrm{ADC}} &= \left\{\sqrt{s}|0\rangle\langle 0| + |1\rangle\langle 1|,\ \sqrt{p}|1\rangle\langle 0|\right\}, && (8)\\
K^{\mathrm{PDC}} &= \left\{\sqrt{s}I,\ \sqrt{p}|0\rangle\langle 0|,\ \sqrt{p}|1\rangle\langle 1|\right\}, && (9)\\
K^{\mathrm{DPC}} &= \{\frac{1}{2}\sqrt{1+3s}\,I,\ \frac{1}{2}\sqrt{p}\,\sigma_x,\ \frac{1}{2}\sqrt{p}\,\sigma_y,\ \frac{1}{2}\sqrt{p}\,\sigma_z\}, && (10)
\end{aligned}$$

with $s \equiv 1-p$. Here the real parameter $p \in [0,1]$ may be time-dependent in some realistic setup [41, 42]. For instance, for the PDC, the parameter s may be like $\exp(-\gamma t)$ with γ the rate of damping.

From Eqs. (6), (8), (9), and (10), the transmission matrix M of each channel can be got through the transformation of the Pauli matrices in the Heisenberg picture [40] as

$$M_{\mathrm{ADC}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{s} & 0 & 0 \\ 0 & 0 & \sqrt{s} & 0 \\ -p & 0 & 0 & s \end{bmatrix}, \quad M_{\mathrm{PDC}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_{\mathrm{DPC}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix}. \tag{11}$$

For simplicity, here we first take as the input states of two-qubit system the Bell diagonal states [5, 12]

$$\rho = \frac{1}{4}\left(I \otimes I + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i \right), \tag{12}$$

which includes the Werner states ($|c_1| = |c_2| = |c_3| = c$) and Bell states ($|c_1| = |c_2| = |c_3| = 1$). This state is physical if the vector (c_1, c_2, c_3) belongs to the tetrahedron defined by the set of the vertices $(-1,-1,-1)$, $(-1,1,1)$, $(1,-1,1)$ and $(1,1,-1)$ [43]. This restriction can be described by the following conditions [43, 5]:

$$\begin{aligned} \sum_{i=1}^{3} c_i &\in [-3, 1], \\ c_i - c_j - c_k &\in [-3, 1] \text{ for } i \neq j \neq k. \end{aligned} \tag{13}$$

For states (12), $\mathcal{R}_0 = \mathrm{diag}\{1, c_1, c_2, c_3\}$ is of diagonal form. From the relation $\mathcal{R} = M\mathcal{R}_0 M^T$, we get $\mathcal{R}$ under the ADC, the PDC, the DPC respectively:

$$\mathcal{R}_{\mathrm{ADC}} = \begin{bmatrix} 1 & 0 & 0 & -p \\ 0 & c_1 s & 0 & 0 \\ 0 & 0 & c_2 s & 0 \\ -p & 0 & 0 & c_3 s^2 + p^2 \end{bmatrix}, \tag{14}$$

$$\mathcal{R}_{\mathrm{PDC}} = \mathrm{diag}\left\{1, c_1 s^2, c_2 s^2, c_3\right\}, \tag{15}$$

$$\mathcal{R}_{\mathrm{DPC}} = \mathrm{diag}\left\{1, c_1 s^2, c_2 s^2, c_3 s^2\right\}. \tag{16}$$

$\mathcal{R}'$ is obtained by deleting the first row of the matrix $\mathcal{R}$, for ADC, PDC, DPC respectively.

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