Assisted distillation of quantum coherence

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We introduce and study the task of assisted coherence distillation. This task arises naturally in bipartite systems where both parties work together to generate the maximal possible coherence on one of the subsystems. Only incoherent operations are allowed on the target system while general local quantum operations are permitted on the other, an operational paradigm that we call local quantum-incoherent operations and classical communication (LQICC). We show that the asymptotic rate of assisted coherence distillation for pure states is equal to the coherence of assistance, a direct analog of the entanglement of assistance, whose properties we characterize. Our findings imply a novel interpretation of the von Neumann entropy: it quantifies the maximum amount of extra quantum coherence a system can gain when receiving assistance from a collaborative party. Our results are generalized to coherence localization in a multipartite setting and possible applications are discussed.

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Introduction. Quantum coherence represents a basic feature of quantum systems that is not present in the classical world. Recently, researchers have begun developing a resource-theoretic framework for understanding quantum coherence [1–9]. In this setting, coherence is regarded as a precious resource that cannot be generated or increased under a restricted class of operations known as incoherent operations [2, 3]. A resource-theoretic treatment of coherence is physically motivated, in part, by certain processes in biology [10–12], transport theory [2, 13, 14], and thermodynamics [7, 15, 16], for which the presence of quantum coherence plays an important role.

In this paper, we consider the task of *assisted coherence distillation*. It involves (at least) two parties, Alice (A) and Bob (B), who share one or many copies of some bipartite state ρ^{AB}. Their goal is to maximize the quantum coherence of Bob's system by Alice performing arbitrary quantum operations on her subsystem, while Bob is restricted to just incoherent operations on his. The duo is further allowed to communicate classically with one another. Overall, we refer to the allowed set of operations in this protocol as ***L**ocal **Q**uantum-**I**ncoherent operations and **C**lassical **C**ommunication (LQICC)*. As we will show, the operational LQICC setting reveals fundamental properties about the quantum coherence accessible to Bob. In particular, the von Neumann entropy of his state, $S(\rho^B)$, quantifies precisely how much extra coherence can be generated in Bob's subsystem using LQICC than when no communication is allowed between him and any correlated party.

Alice and Bob's objective here is analogous to the task of assisted *entanglement* distillation. In the latter, entanglement is shared between three parties, A, B, C, and the goal is for B and C to obtain maximal bipartite entanglement when all parties use (unrestricted) Local Operations and Classical Communication (LOCC). The corresponding maximal entanglement that can be generated between B and C is known as "entanglement of collaboration" [17]. By direct analogy, we define the "coherence of collaboration" as the maximum coherence that can be generated on subsystem B by LQICC operations. In general, both LOCC and LQICC protocols can be very complicated, involving many multiple rounds of measurement and communication [18]. A simplified scenario considers one-way protocols in which Alice holds a purifying system, and only she is allowed to broadcast measurement data. The maximum entanglement for B and C (resp. maximum coherence for B) that can be generated in this manner is called the "entanglement of assistance" [19] (resp. will be called the "coherence of assistance"). In the asymptotic setting the entanglement of assistance is known to be equal to the entanglement of collaboration if the overall state is pure [20]. We show an equivalent result for coherence: for pure states the coherence of assistance is equal to the coherence of collaboration in the asymptotic setting, and a closed expression for these quantities is also provided. Moreover, when Bob's system is a qubit and the overall state is pure, the coherence of assistance and the coherence of collaboration are equivalent even in the single-copy case. Finally, we also present a generalization to a multipartite setting where many assisting players collaborate to localize coherence onto a target system, and discuss possible applications to quantum technologies.

Resource theory of coherence. The starting point of our work is the resource theory of coherence, introduced recently in [2–4, 8]. In particular, a quantum state ρ is said to be incoherent in a given reference basis $\{|i\rangle\}$, if the state is diagonal in this basis, i.e., if $\rho = \sum_i p_i |i\rangle\langle i|$ with some probabilities p_i. For a bipartite system, the reference basis is assumed to be a tensor product of local bases [4, 5, 8].

A quantum operation is said to be incoherent, if each of its Kraus operators K_α is incoherent, i.e., if $K_\alpha \mathcal{I} K_\alpha^\dagger \subseteq \mathcal{I}$, where $\mathcal{I}$ is the set of incoherent states. Completely dephasing any state ρ in the incoherent basis will generate the incoherent state $\Delta(\rho) := \sum_i q_i|i\rangle\langle i|$ with $q_i = \langle i|\rho|i\rangle$. If d is the dimension of the Hilbert space of the system, the maximally coherent state is $|\Phi_d\rangle = \sqrt{1/d}\sum_i |i\rangle$, and we let $|\Phi\rangle := |\Phi_2\rangle$ denote the "unit" coherence resource state [3].

Similar to the framework of entanglement distillation [21, 22], general quantum states can be used for asymptotic distillation of maximally coherent states via incoherent operations. Even more, a closed expression for the optimal distillation rate was found recently by Winter and Yang [8], and turns out to be equal to the relative entropy of coherence introduced in [1, 3].

Lemma 1. *The distillable coherence of ρ is [8]*

$$C_d(\rho) = C_r(\rho) = S(\Delta(\rho)) - S(\rho), \tag{1}$$

where $C_r(\rho)$ is the relative entropy of coherence, defined as $C_r(\rho) = \min_{\sigma\in\mathcal{I}} S(\rho\|\sigma)$, and $S(\rho\|\sigma) = -tr(\rho\log\sigma) - S(\rho)$.

Note that $C_d(\rho) > 0$ if and only if ρ is not incoherent.

Coherence of collaboration. We now move to the main topic of this work, namely the assisted distillation of coherence. As mentioned earlier, in this setting two parties Alice and Bob share many copies of a joint state $\rho = \rho^{AB}$ and aim to maximize coherence on Bob's system by LQICC operations.

In order to make a quantitative analysis, we define the *distillable coherence of collaboration* as the optimal rate, i.e., the optimal number of maximally coherent states on Bob's side per copy of the shared resource state ρ, in the assisted setting:

$$C_d^{A|B}(\rho) = \sup\left\{R : \lim_{n\to\infty}\left(\inf_{\Lambda}\left\|\Lambda\left[\rho^{\otimes n}\right] - \Phi^{\otimes\lfloor Rn\rfloor}\right\|\right) = 0\right\}. \tag{2}$$

Here, $\|M\| = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm, and the infimum is taken over all LQICC operations Λ. When Alice is uncorrelated from Bob, i.e. $\rho^{AB} = \rho^A\otimes\rho^B$, then $C_d^{A|B}(\rho^{AB})$ reduces to the distillable coherence $C_d(\rho^B)$ which can be evaluated exactly using Lemma 1 [8]. In the following, we are interested in understanding how the assistance of Alice can improve Bob's distillation rate, i.e., how larger $C_d^{A|B}(\rho^{AB})$ can be in comparison to $C_d(\rho^B)$. For answering this question, we first note that the set of bipartite states which can be created via LQICC operations, that will be referred to as the set $\mathcal{QI}$ of *quantum-incoherent* (QI) states, admits a simple characterization. Namely, all such states have the following form:

$$\chi^{AB} = \sum_i p_i\sigma_i^A\otimes|i\rangle\langle i|^B. \tag{3}$$

Here, σ_i^A are arbitrary quantum states on A, and the states $|i\rangle^B$ belong to the local incoherent basis of B. Note that QI states have the same form as general quantum-classical states [23] (i.e., states with vanishing quantum discord [24]), except the "classical" part must be diagonal in the fixed incoherent basis.

It is obvious that any QI state has $C_d^{A|B}(\rho^{AB}) = 0$, and the following theorem shows that the converse is true as well.

Theorem 2. *A state ρ^{AB} has $C_d^{A|B}(\rho^{AB}) > 0$ if and only if the state ρ^{AB} is not quantum-incoherent.*

This theorem shows that any state which cannot be created for free via LQICC operations constitutes a resource for extracting coherence on Bob's side. For the proof of the theorem we refer to the Supplemental Material.

In the next step, we will provide an upper bound on the distillable coherence of collaboration. For this, we introduce the QI relative entropy:

$$C_r^{A|B}(\rho^{AB}) = \min_{\chi^{AB}\in\mathcal{QI}} S(\rho^{AB}\|\chi^{AB}) \tag{4}$$

with the minimization taken over the set of QI states. It is in order to note that $C_r^{A|B}$ is different from the relative entropy of discord introduced in [25, 26], as the latter involves a minimization over all bases of B, while Eq. (4) is defined for a fixed incoherent basis $\{|i\rangle^B\}$. Using the same reasoning as in [26, see Theorem 2 there], it is straightforward to see that $C_r^{A|B}$ can also be written as

$$C_r^{A|B}(\rho^{AB}) = S\left(\Delta^B(\rho^{AB})\right) - S(\rho^{AB}) \tag{5}$$

with $\Delta^B(\rho^{AB}) := \sum_i(\mathbb{I}\otimes|i\rangle\langle i|)\rho^{AB}(\mathbb{I}\otimes|i\rangle\langle i|)$. Moreover, since the relative entropy does not increase under general quantum operations, $C_r^{A|B}$ is monotonically nonincreasing under LQICC operations. The following theorem shows that the QI relative entropy is an upper bound on $C_d^{A|B}$.

Theorem 3. *Given a state ρ^{AB} shared by Alice and Bob, the distillable coherence of collaboration is bounded above according to*

$$C_d^{A|B}(\rho^{AB}) \leq C_r^{A|B}(\rho^{AB}). \tag{6}$$

The proof of the theorem can be found in the Supplemental Material. This result shows that in the task considered here the relative entropy plays similar role as in the task of entanglement distillation [27], bounding the distillation rate from above. Note that for standard coherence distillation the relative entropy of coherence is indeed equal to the optimal distillation rate [8], see also Lemma 1. It is an open question if this is also true for the task considered here, i.e., if the inequality (6) is an equality for all quantum states ρ^{AB}. As we will see in Theorem 4 below, at least for pure states the answer is affirmative.

Coherence of assistance. We now introduce the *coherence of assistance* (CoA) for a state ρ as the maximal average coherence of the state:

$$C_a(\rho) = \max\sum_i q_iC_r(\psi_i) = \max\sum_i q_iS(\Delta(\psi_i)), \tag{7}$$

where the maximization is taken over all pure-state decompositions of $\rho = \sum_i q_i|\psi_i\rangle\langle\psi_i|$, and ψ_i is denoting $|\psi_i\rangle\langle\psi_i|$.

To provide CoA with an operational interpretation it is instrumental to compare it with entanglement of assistance (EoA) originally proposed by DiVincenzo *et al.* [19]. For a bipartite state ρ^{BC}, one identifies a decomposition of maximal average entanglement:

$$E_a(\rho^{BC}) = \max\sum_i q_iE(\psi_i^{BC}) = \max\sum_i q_iS(tr_B\psi^{BC}), \tag{8}$$

for $\rho^{BC} = \sum_i q_i|\psi_i\rangle\langle\psi_i|^{BC}$. The interpretation of EoA is that by using local measurement and one-way classical communication, Alice can help Bob and Charlie obtain an average entanglement of at most $E_a(\rho^{BC})$ when they all share $|\Psi\rangle^{ABC}$, a purification of ρ^{BC}. In this case, any possible pure-state decomposition of ρ^{BC} can be realized when Alice performs a suitable measurement and announces the result [28]. If all the parties have access to arbitrary number of copies of the total state $|\Psi\rangle^{ABC}$, the figure of merit is the regularized EoA $E_a^\infty(\rho) = \lim_{n\to\infty}\frac{1}{n}E_a(\rho^{\otimes n})$. For an arbitrary density matrix ρ^{BC}, the regularized EoA is simply given by [20]

$$E_a^\infty(\rho^{BC}) = \min\{S(\rho^B), S(\rho^C)\}. \tag{9}$$

Similarly, the CoA defined in Eq. (7) has an operational meaning if we assume that the state $\rho = \rho^B$ belongs to Bob, who is assisted by another party (Alice) holding a purification of ρ^B. Together with Lemma 1 then, $C_a(\rho^B)$ quantifies a one-way coherence distillation rate for Bob when Alice applies the same procedure for each copy of the state. In the many-copy setting, higher one-way distillation rates can typically be obtained when Alice performs a joint measurement across her many copies. Thus, in direct analogy to EoA, we consider the regularized CoA defined as $C_a^\infty(\rho) := \lim_{n\to\infty}\frac{1}{n}C_a(\rho^{\otimes n})$.

As we prove in the Supplemental Material, the CoA of a state $\rho = \sum_{i,j}\rho_{ij}\,|i\rangle\langle j|$ is equal to the EoA of the corresponding maximally correlated state [29] $\rho_{\mathrm{mc}} = \sum_{i,j}\rho_{ij}\,|ii\rangle\langle jj|$:

$$C_a(\rho) = E_a(\rho_{\mathrm{mc}}). \tag{10}$$

Clearly, Eq. (10) implies that this equality is also true for the regularized quantities: $C_a^\infty(\rho) = E_a^\infty(\rho_{\mathrm{mc}})$. Using Eq. (9), the regularized CoA thus acquires the simple expression:

$$C_a^\infty(\rho) = S\left(\Delta(\rho)\right). \tag{11}$$

Equipped with these tools we are now in position to provide a closed expression for $C_d^{A|B}$ for all pure states.

Theorem 4. *For a pure state $|\Psi\rangle^{AB}$ shared by Alice and Bob, the following equality holds:*

$$C_d^{A|B}(|\Psi\rangle^{AB}) = C_a^\infty(\rho^B) = C_r^{A|B}(|\Psi\rangle^{AB}) = S(\Delta(\rho^B)). \tag{12}$$

The proof of the theorem can be found in the Supplemental Material. With Theorem 4 in hand, we give the von Neumann entropy an alternative operational interpretation. Namely, let $\delta C_d(\rho^B)$ denote the maximal increase in distillable coherence that Bob can obtain when exchanging classical communication with a correlated party; i.e. $\delta C_d(\rho^B) = \max_{\rho^{AB}}[C_d^{A|B}(\rho^{AB}) - C_d(\rho^B)]$, where the maximization is taken over all extensions ρ^{AB} of ρ^B. Noticing that the maximum is attained if ρ^{AB} is pure, Lemma 1 and Theorem 4 imply that

$$\delta C_d(\rho^B) = S(\rho^B). \tag{13}$$

Interestingly, this result does not depend on the particular choice of the reference incoherent basis.

Let us turn to the obvious inequality $C_a(\rho^B) \leq C_a^\infty(\rho^B)$ and ask whether C_a is additive, in which case the inequality becomes tight. This question is especially interesting when one considers Ref. [8] where the *coherence of formation*, defined with a minimization rather than a maximization in Eq. (7), and thus a dual quantity to the CoA, is shown to be additive. Below, we will show that in contrast, CoA fails to exhibit additivity in general. Nevertheless, when restricting attention to n copies of an arbitrary single-qubit state ρ, additivity of CoA can be proven. The latter finding is quite noteworthy since no analogous result is known for EoA in two-qubit systems.

Theorem 5. *CoA is n-copy additive for qubit states ρ:*

$$C_a(\rho) = C_a^\infty(\rho) = S(\Delta(\rho)). \tag{14}$$

However, in general the CoA is not additive.

We refer to the Supplemental Material for the proof. It is interesting to note that we prove non-additivity for systems with dimension 4 and above. Thus, it remains open if C_a is additive for qutrits. Note that by Theorem 4, this result implies that optimal coherence distillation for single-qubit systems involves just one-way communication and single-copy measurements from a purifying auxiliary system.

***Multipartite scenario*.** We now extend our results to the multipartite setting. When more than one party is providing assistance, the process of collaboratively generating coherence for Bob's system will be called *coherence localization*, in direct analogy to the task of *entanglement localization* [30].

We consider $(N+1)$-partite states $\rho^{A_1\cdots A_N B}$, where the parties $A_1, \cdots, A_N$ are allowed to perform arbitrary local quantum operations, and the party B is restricted to incoherent operations only. Additionally, classical communication is allowed between all the parties. The aim of all the parties is to localize as much coherence as possible on the subsystem of B. The corresponding asymptotic coherence localization rate can be defined just as in Eq. (2) and will be denoted by $C_d^{A_1,\cdots,A_N|B}(\rho^{A_1\cdots A_N B})$. For total pure states with B being a qubit we find that, quite remarkably, individual measurements on the auxiliary systems can generate the same maximal coherence for the target system B as when a global measurement is performed across all the auxiliary systems $A_1, \cdots, A_N$.

Theorem 6. *Let $|\Psi\rangle^{A_1,\cdots,A_N B}$ be an arbitrary multipartite state with system B being a qubit. Then*

$$C_d^{A_1,\cdots,A_N|B}\left(|\Psi\rangle^{A_1,\cdots,A_N B}\right) = C_d^{A_{tot}|B}\left(|\Psi\rangle^{A_{tot}B}\right) = S\left(\Delta(\rho^B)\right), \tag{15}$$

where $A_{tot} = A_1, \cdots, A_N$ is viewed as one party with the locality constraint removed among the A_i.

The proof is deferred to the Supplemental Material. This theorem implies that for asymptotic coherence localization the assisting parties $A_1, \cdots, A_N$ do not need access to a quantum channel: local quantum operations on their subsystems together with classical communication are enough to ensure maximal coherence localization. This is true if the total state is pure, and if coherence is localized on a qubit.

LQICC versus SLOCC protocols. The proof of Theorem 4 relied on relating the tasks of assisted coherence distillation and assisted entanglement distillation. This further supports a conjecture put forth in Ref. [8] that the resource theory of coherence is equivalent to the resource theory of entanglement for maximally correlated states [29]. We can prove a more general connection between LQICC operations in the coherence setting and LOCC operations in the entanglement setting.

For a given bipartite state ρ^{AB} we define the association

$$\rho^{AB} = \sum_{ij} M_{ij}^{A} \otimes |i\rangle\langle j|^{B} \Rightarrow \tilde{\rho}^{ABC} = \sum_{ij} M_{ij}^{A} \otimes |ii\rangle\langle jj|^{BC}, \quad (16)$$

where M_{ij} are operators acting on Alice's space and $\{|i\rangle\}$ is the fixed incoherent basis. As we show in the Supplemental Material, if two states ρ^{AB} and σ^{AB} are related via a bipartite LQICC map, i.e. $\sigma^{AB} = \Lambda_{\mathrm{LQICC}}[\rho^{AB}]$, then the corresponding states $\tilde{\rho}^{ABC}$ and $\tilde{\sigma}^{ABC}$ are related via a tripartite stochastic LOCC (SLOCC) map, i.e. $\tilde{\sigma}^{ABC} = \Lambda_{\mathrm{SLOCC}}[\tilde{\rho}^{ABC}]$. Therefore any procedure which can be implemented "for free" in the framework of assisted coherence has an equivalent probabilistic "free" implementation on the level of maximally correlated states. We find that, in fact, for many LQICC transformations $\rho^{AB} \to \sigma^{AB}$, the corresponding LOCC transformation $\tilde{\rho}^{ABC} \to \tilde{\sigma}^{ABC}$ can be implemented with probability one. It is an interesting open question whether the (tripartite) LOCC analog to every (bipartite) LQICC transformation has always a deterministic implementation.

In the case where the subsystem A is uncorrelated, Eq. (16) reduces to $\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \Rightarrow \rho_{\mathrm{mc}} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|$. For this situation, the above results imply that for any two states ρ and $\sigma = \Lambda_{\mathrm{i}}[\rho]$ related via an incoherent operation Λ_{i}, the corresponding maximally correlated states ρ_{mc} and σ_{mc} are related via bipartite SLOCC: $\sigma_{\mathrm{mc}} = \Lambda_{\mathrm{SLOCC}}[\rho_{\mathrm{mc}}]$. Moreover, in the asymptotic setting where many copies of ρ are available, the SLOCC procedure becomes deterministic whenever the entanglement cost of σ_{mc} is not larger than the distillable entanglement of ρ_{mc}. This criterion can be easily checked, recalling that for these states the entanglement cost is equal to the entanglement of formation [31, 32], and their distillable entanglement admits a simple expression [29].

Conclusions. In conclusion, based on the resource theory of coherence formalized by Baumgratz *et al.* [3], we introduced the framework of assisted coherence distillation and, more generally, of coherence localization. In this framework, in a multipartite system, one party aims to distill coherence via incoherent operations on a target subsystem, with the aid of the other parties who have access to classical communication channels and can act with arbitrary local quantum operations on their subsystems. In the bipartite setting (one target and one collaborative system), we characterized all states which can be used for assisted coherence distillation in this procedure, and showed that the maximal amount of coherence distillable in this way is bounded above by the relative entropy distance from the set of free states which cannot be created by the allowed operations. For pure states, the amount of distillable coherence of collaboration was shown to be equal to the asymptotic coherence of assistance of the incoherent party, a quantity dual to the coherence of formation, and closely related to the entanglement of assistance. The maximum extra amount of coherence that one party can distill with *viz.* without the assistance of an external player is given exactly by the von Neumann entropy of the target system. Therefore, the more mixed a state is, the more it benefits from collaboration for the task of extracting coherence. In the many-qubit scenario, we showed that, for pure shared states, collective operations on all the collaborative parties are not required for optimal coherence localization onto a target qubit.

There are many scenarios of practical relevance where the task of assisted coherence distillation, or localization, can play a central role. For instance, we can think of a remote or not entirely accessible system on which coherence is needed as a resource (e.g. a biological system): our results give optimal prescriptions to inject such coherence on the remote target by acting on a controllable ancilla. An assisted distillation procedure is also useful in order to maximize the thermodynamical work extraction from the coherence of the target system [7]. In a multipartite setting, one can imagine to distribute a correlated state among many parties, and implement an instance of open-destination quantum metrology (a protocol somehow inspired by open-destination teleportation [33]), in which one party is selected to estimate an unknown parameter [34] and the other parties act locally on their subsystems in order to localize as much coherence as possible on the chosen target, so as to enhance the estimation precision. Similarly, the task can be a useful primitive within a secure quantum cryptographic network [35], in which the distribution of non-orthogonal states (and thus coherence) is required. These and other collaborative games will be investigated in further work.

From an information-theoretic point of view [36, 37], our study supports the intriguing conjecture that the theory of coherence is equivalent to the theory of entanglement of maximally correlated states [5, 8]. We also remark that the theory of (assisted) coherence is not a fully reversible one. In reversible quantum resource theories [37], if a resource state ρ can be asymptotically converted to another resource state σ at rate R with free operations, then σ can be converted back to ρ at rate $1/R$. However, note that in our case the maximally entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$ can be converted to the state $|0+\rangle$ via LQICC operations at rate 1, but the process is not reversible as LQICC operations cannot produce entanglement.

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SUPPLEMENTAL MATERIAL

Proof of Theorem 2

Here we will prove that any state which is not quantum-incoherent (QI) has nonzero distillable coherence of collaboration $C_d^{A|B}(\rho^{AB}) > 0$. To prove this, suppose that ρ^{AB} is not QI. We can always expand

$$\rho^{AB} = \sum_{i,j} |e_i\rangle\langle e_j|^A \otimes N_{ij}^B, \tag{A.1}$$

where the $|e_i\rangle^A$ form an orthonormal basis for Alice's Hilbert space and the N_{ij}^B are some operators on Bob's space. Note that the operators N_{ii}^B are nonnegative, i.e., $N_{ii}^B \geq 0$, and can be written as $q_i\rho_i^B$. The state ρ_i^B can be seen as the post-measurement state of Bob if Alice performs a von Neumann measurement in the basis $|e_i\rangle^A$, and q_i is the corresponding probability. If for some outcome i with nonzero probability $q_i > 0$ the corresponding state ρ_i^B is not incoherent, then Lemma 1 in the main text guarantees that $C_d^{A|B}(\rho^{AB}) \geq q_i C_r(\rho_i^B) > 0$.

In the next step, we will consider the case where all the states ρ_i^B corresponding to nonzero outcome probability $q_i > 0$ are incoherent (i.e. all the operators N_{ii} are diagonal w.r.t. the incoherent basis). Then, the condition that the state ρ^{AB} is not QI implies that N_{kl} must have off-diagonal elements for some $k \neq l$. Using the fact that $N_{kl} = N_{lk}^\dagger$, we see that at least one of the operators $N_{kl} + N_{kl}^\dagger$ or $i(N_{kl} - N_{kl}^\dagger)$ must also contain offdiagonal elements in this case. Depending on what is the case, Alice performs a von Neumann measurement in a basis containing the state $\cos\theta|e_k\rangle^A + \sin\theta|e_l\rangle^A$ or in a basis containing the state $\cos\theta|e_k\rangle^A + i\sin\theta|e_l\rangle^A$ with some angle θ which will be determined below. In the first case the post-measurement state of Bob ρ_θ^B is given by

$$p_\theta\rho_\theta^B = \cos^2\theta N_{kk} + \sin^2\theta N_{ll} + \cos\theta\sin\theta(N_{kl} + N_{lk}), \tag{A.2}$$

where p_θ is the corresponding outcome probability. Since $\cos^2\theta$, $\sin^2\theta$, and $\cos\theta\sin\theta$ are linearly independent, the trace of the right-hand side of Eq. (A.2) cannot vanish for all θ. Hence, there are some $0 < \theta < \pi/2$ for which $p_\theta > 0$. Similarly, in the second case the post-measurement state of Bob σ_θ^B is given by

$$q_\theta\sigma_\theta^B = \cos^2\theta N_{kk} + \sin^2\theta N_{ll} + i\cos\theta\sin\theta(N_{kl} - N_{lk}) \tag{A.3}$$

with outcome probability q_θ. By the same argument, there are some $0 < \theta < \pi/2$ for which $q_\theta > 0$. Moreover, in both of the above cases the post-measurement state of Bob contains offdiagonal elements.

Finally, we will now show how the above results imply that $C_d^{A|B}(\rho^{AB}) > 0$ is true for any state which is not QI. In particular, we proved that for any such state Alice can perform a local von Neumann measurement in such a way that the post-measurement state of Bob contains nonzero coherence with nonvanishing probability. This means that by repeating this procedure on each copy of ρ^{AB}, Bob will end up with many copies of a state having nonzero coherence. Then, by using Lemma 1 from the main text Bob can distill maximally coherent states with nonzero rate. This completes the proof of the theorem.

Proof of Theorem 3

In the following, we will prove that for any state ρ^{AB} the distillable coherence of collaboration $C_d^{A|B}$ is bounded above by the QI relative entropy $C_r^{A|B}$:

$$C_d^{A|B}(\rho^{AB}) \leq C_r^{A|B}(\rho^{AB}). \tag{A.4}$$

To prove this statement, we first note that C_d can also be expressed as follows:

$$C_d^{A|B}(\rho^{AB}) = \sup\left\{C_r(|\phi\rangle) : \lim_{n\to\infty}\left(\inf_\Lambda\left\|\Lambda\left[\rho_i^{\otimes n}\right] - \rho_f^{\otimes n}\right\|\right) = 0\right\}, \tag{A.5}$$

with the initial state $\rho_i = \rho^{AB} \otimes |0\rangle\langle 0|^{\tilde{B}}$, the final state $\rho_f = |00\rangle\langle 00|^{AB} \otimes |\phi\rangle\langle\phi|^{\tilde{B}}$, $\tilde{B}$ is an additional particle in Bob's hands, and the infimum in Eq. (A.5) is taken over all LQICC operations Λ between Alice and Bob.

Then, by definition of $C_d^{A|B}$ in Eq. (A.5), for any $\varepsilon > 0$ there exists a state $|\phi\rangle$, an integer n, and an LQICC protocol Λ_n acting on n copies of the state ρ_i such that

$$C_d^{A|B}(\rho^{AB}) - C_r(|\phi\rangle) \leq \varepsilon, \tag{A.6}$$

$$\left\|\Lambda_n\left[\rho_i^{\otimes n}\right] - \rho_f^{\otimes n}\right\| \leq \varepsilon. \tag{A.7}$$

In the next step, we will prove continuity of C_r. In particular for two states ρ^{XY} and σ^{XY} with $\|\rho^{XY} - \sigma^{XY}\| \leq 1$ the QI relative entropy C_r is continuous in the following sense:

$$|C_r^{X|Y}(\rho^{XY}) - C_r^{X|Y}(\sigma^{XY})| \leq 2T\log_2 d_{XY} + 2h(T), \tag{A.8}$$

where $T = \|\rho^{XY} - \sigma^{XY}\|/2$ is the trace distance, d_{XY} is the dimension of the total system, and

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x) \tag{A.9}$$

is the binary entropy. It is straightforward to prove Eq. (A.8) by using continuity of the von Neumann entropy [38].

The continuity relation in Eq. (A.8) together with Eq. (A.7) implies that for any $0 < \varepsilon \leq 1/2$ there exists an integer $n \geq 1$ and an LQICC protocol Λ_n acting on n copies of the state ρ_i such that

$$C_r^{A|B\tilde{B}}(\Lambda_n[\rho_i^{\otimes n}]) \geq C_r^{A|B\tilde{B}}(\rho_f^{\otimes n}) - 2n\varepsilon\log_2 d - 2h(\varepsilon), \tag{A.10}$$

where d is the dimension of the total system $AB\tilde{B}$. Since the QI relative entropy C_r is additive and does not increase under LQICC operations, it follows that for any $0 < \varepsilon \leq 1/2$ there exists an integer $n \geq 1$ such that

$$C_r^{A|B\tilde{B}}(\rho_i) \geq C_r^{A|B\tilde{B}}(\rho_f) - 2\varepsilon\log_2 d - \frac{2}{n}h(\varepsilon). \tag{A.11}$$

By using the relations $C_r^{A|B\tilde{B}}(\rho_i) = C_r^{A|B}(\rho^{AB})$ and $C_r^{A|B\tilde{B}}(\rho_f) = C_r(|\phi\rangle)$, the latter inequality implies

$$C_r^{A|B}(\rho^{AB}) \geq C_r(|\phi\rangle). \tag{A.12}$$

On the other hand, Eq. (A.6) means that $C_r(|\phi\rangle) \geq C_d^{A|B}(\rho^{AB}) - \varepsilon$. Combining these results completes the proof of the theorem.

Coherence of assistance and entanglement of assistance of maximally correlated states

In the following we will prove the relation

$$C_a(\rho) = E_a(\rho_{\mathrm{mc}}), \tag{A.13}$$

where the state $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$ is arbitrary, and the state $\rho_{\mathrm{mc}} = \sum_{i,j} \rho_{ij} |ii\rangle\langle jj|$ is the maximally correlated state associated with ρ.

For proving Eq. (A.13), consider an optimal decomposition of the state $\rho_{\mathrm{mc}} = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ such that

$$E_a(\rho_{\mathrm{mc}}) = \sum_k p_k E(|\psi_k\rangle), \tag{A.14}$$

where the entanglement of a pure state $|\psi\rangle^{XY}$ is given by the von Neumann entropy of the reduced state: $E(|\psi\rangle^{XY}) = S(\rho^X)$. Note that every state $|\psi_k\rangle$ in the above decomposition can be written in the form $|\psi_k\rangle = \sum_i c_i^k |ii\rangle$ with complex coefficients c_i^k [32]. In the next step, we introduce states $|\phi_k\rangle = \sum_i c_i^k |i\rangle$, and note that together with probabilities p_k these states give rise to a decomposition of the state $\rho = \sum_k p_k |\phi_k\rangle\langle\phi_k|$. Note that this decomposition of ρ is optimal for the coherence of assistance:

$$C_a(\rho) = \sum_k p_k C_r(|\phi_k\rangle). \tag{A.15}$$

The proof of Eq. (A.13) is complete by using the relation $C_r(|\phi_k\rangle) = E(|\psi_k\rangle)$.

Proof of Theorem 4

Proof. In the following we will prove the equality

$$C_d^{A|B}(|\Psi\rangle^{AB}) = C_a^{\infty}(\rho^B) = C_r^{A|B}(|\Psi\rangle^{AB}) = S(\Delta(\rho^B)). \tag{A.16}$$

Clearly, the regularized CoA of a state $\rho^B = tr_A|\Psi\rangle\langle\Psi|^{AB}$ cannot be larger than $C_d^{A|B}$ of its purification:

$$C_a^{\infty}(\rho^B) \leq C_d^{A|B}(|\Psi\rangle^{AB}). \tag{A.17}$$

Together with Eq. (11) in the main text one obtains the lower bound

$$S(\Delta(\rho^B)) \leq C_d^{A|B}(|\Psi\rangle^{AB}). \tag{A.18}$$

On the other hand, Eq. (5) in the main text implies

$$C_r^{A|B}(|\Psi\rangle^{AB}) = S(\Delta(\rho^B)). \tag{A.19}$$

Together with Theorem 3 this completes the proof. □

Proof of Theorem 5

In the following, we will prove the equality

$$C_a(\rho) = C_a^{\infty}(\rho) = S(\Delta(\rho)) \tag{A.20}$$

for any single-qubit state ρ.

Let $|\Psi\rangle^{AB}$ be an arbitrary purification for ρ^B, and expand in the incoherent basis as

$$|\Psi\rangle^{AB} = \sum_{k=0}^{1} \sqrt{p_k}\, |\psi_k\rangle^A |k\rangle^B, \tag{A.21}$$

where $|\psi_k\rangle^A$ are arbitrary states for Alice. In the next step we note that there always exist orthogonal states $|\eta_\pm\rangle^A$ which form a mutually unbiased basis with respect to the two states $|\psi_k\rangle^A$. Thus, the states $|\psi_k\rangle^A$ can be written as

$$|\psi_k\rangle^A = \frac{1}{\sqrt{2}}(e^{i\alpha_k} |\eta_+\rangle^A + e^{i\beta_k} |\eta_-\rangle^A) \tag{A.22}$$

with some reals α_k and β_k.

When Alice performs a von Neumann measurement in the $|\eta_\pm\rangle^A$ basis, Bob will find his system in one of the post-measurement states

$$|\phi_\pm\rangle^B = \sqrt{p_0} e^{i\vartheta_\pm} |0\rangle^B + \sqrt{p_1} e^{i\varphi_\pm} |1\rangle^B \tag{A.23}$$

with some reals $\vartheta_\pm$ and $\varphi_\pm$ for the +/- outcome respectively. In both cases, the state has coherence $C_r(|\phi_\pm\rangle^B) = S(\Delta(\rho^B))$. The above reasoning shows that $C_a(\rho) = S(\Delta(\rho))$ is true for any single-qubit state ρ. Recalling that $C_a^{\infty}(\rho) = S(\Delta(\rho))$ is true for any quantum state ρ, the proof of Eq. (A.20) is complete.

We will now show that there exist states ρ of dimension 4 such that

$$C_a(\rho) < C_a^{\infty}(\rho). \tag{A.24}$$

This inequality also implies that the coherence of assistance cannot be additive. For proving this, consider the $2 \otimes 4$ state

$$|\Psi\rangle^{AB} = \frac{1}{2}\left(|00\rangle + |11\rangle + |{+}2\rangle + |\hat{+}3\rangle\right) \tag{A.25}$$

with $|\hat{+}\rangle = 1/\sqrt{2}(|+\rangle + i|1\rangle)$. We will show that the reduced state ρ^B satisfies $C_a(\rho^B) < C_a^{\infty}(\rho^B) = 2$. We will prove this by showing a slightly stronger statement: for any measurement of Alice performed on the state in Eq. (A.25), the corresponding post-measurement state of Bob will have coherence strictly below 2.

This can be seen by contradiction: assume that for some measurement of Alice with POVM element M^A the corresponding post-measurement state of Bob has maximal coherence, i.e. it corresponds to the state $|\Phi_4\rangle = 1/2 \sum_{i=0}^{3} |i\rangle$. This condition can also be written as follows:

$$\mathrm{Tr}_A[M^A |\Psi\rangle\langle\Psi|^{AB}] = p\, |\Phi_4\rangle\langle\Phi_4|^B, \tag{A.26}$$

where $M^A \leq \mathbb{1}^A$ is a nonnegative operator on the subsystem A and $p > 0$ is the probability of Alice's outcome.

In the next step it is crucial to note that Eq. (A.26) can only be fulfilled if M has the same nonzero overlap with all the states $|0\rangle$, $|1\rangle$, $|+\rangle$, and $|\hat{+}\rangle$:

$$\langle 0|M|0\rangle = \langle 1|M|1\rangle = \langle +|M|+\rangle = \langle \hat{+}|M|\hat{+}\rangle > 0. \tag{A.27}$$

Denoting the elements of M by $M_{kl} = \langle k|M|l\rangle$, the above equality leads to

$$\begin{aligned} M_{00} = M_{11} &= \frac{1}{2}\left(M_{00} + M_{11} + M_{01} + M_{10}\right) \\ &= \frac{1}{2}\left(M_{00} + M_{11} + iM_{01} - iM_{10}\right). \end{aligned} \tag{A.28}$$

Taking into account that M is nonnegative, this set of equations has only one solution, namely $M_{00} = M_{11} = M_{01} = M_{10} = 0$. This completes the proof. Interestingly, from the above consideration it is not clear if $C_a(\rho)$ is additive for qutrit states.

Proof of Theorem 6

Here we will prove the equality

$$C_d^{A_1,\cdots,A_N|B}\left(|\Psi\rangle^{A_1,\cdots,A_N B}\right) = C_d^{A_{tot}|B}\left(|\Psi\rangle^{A_{tot}B}\right) = S\left(\Delta(\rho^B)\right), \tag{A.29}$$

where B is a qubit, and $A_{tot} = A_1 \cdots A_N$ denotes the total system except for B. In the following, we assume that the parties $A_1, \ldots, A_N$ can perform arbitrary local operations, the party B is restricted to incoherent operations, and classical communication is allowed between all parties.

For proving this statement, we will show that for some LOCC protocol on $A_1, \ldots, A_N$ all post-measurement states of B will have coherence $S\left(\Delta(\rho^B)\right)$. This means that by Lemma 1 of the main text the state $|\Psi\rangle^{A_1,\cdots,A_N B}$ can be used to extract coherence at rate $S\left(\Delta(\rho^B)\right)$. This will complete the proof, since by Theorem 3 of the main text it is not possible to achieve more coherence on B even by joint operations on $A_1, \ldots, A_N$.

In the following, we will use similar arguments as in the proof of Theorem 5. In the first step, we expand the state $|\Psi\rangle^{A_1,\cdots,A_N B}$ in Bob's incoherent basis, arriving at

$$|\Psi\rangle^{A_1,\cdots,A_N B} = \sum_{k=0}^{1} \sqrt{p_k}\, |\psi_k\rangle^{A_1,\cdots,A_N} |k\rangle^B. \tag{A.30}$$

Similar to the proof of Theorem 5, we note that there exist orthogonal multipartite states $|\eta_\pm\rangle$ which form a mutually unbiased basis with respect to the states $|\psi_k\rangle$. In other words, the states $|\psi_k\rangle$ can be written as

$$|\psi_k\rangle = \frac{1}{\sqrt{2}}(e^{i\alpha_k}|\eta_+\rangle + e^{i\beta_k}|\eta_-\rangle) \tag{A.31}$$

with some reals α_k and β_k.

To complete the proof we will use the results of Walgate *et al.* [39], showing that any two multipartite orthogonal states $|\eta_+\rangle$ and $|\eta_-\rangle$ can be perfectly distinguished via LOCC. Their results also imply the existence of a POVM $\{\Pi_+, \Pi_-\}$ which can be implemented via LOCC such that

$$\Pi_+|\eta_-\rangle = \Pi_-|\eta_+\rangle = 0. \tag{A.32}$$

Applying this POVM on systems $A_1 \cdots A_N$ of the state $|\Psi\rangle^{A_1,\cdots,A_N B}$ will generate post-measurement states for Bob of the form

$$|\phi_\pm\rangle^B = \sqrt{p_0}e^{i\vartheta_\pm}|0\rangle^B + \sqrt{p_1}e^{i\varphi_\pm}|1\rangle^B, \tag{A.33}$$

which leaves him with optimal coherence $C_r(|\phi_\pm^B\rangle) = S(\Delta(\rho^B))$. This completes the proof of the theorem.

Relating LQICC and tripartite SLOCC maps

Here we will prove that for any pair of bipartite states

$$\rho^{AB} = \sum_{i,j} M_{ij}^A \otimes |i\rangle\langle j|^B, \qquad \sigma^{AB} = \sum_{i,j} N_{ij}^A \otimes |i\rangle\langle j|^B$$

related via $\sigma^{AB} = \Lambda_{\mathrm{LQICC}}[\rho^{AB}]$, with an LQICC operation Λ_{LQICC}, the corresponding tripartite states

$$\tilde{\rho}^{ABC} = \sum_{i,j} M_{ij}^A \otimes |ii\rangle\langle jj|^{BC}, \qquad \tilde{\sigma}^{ABC} = \sum_{i,j} N_{ij}^A \otimes |ii\rangle\langle jj|^{BC}$$

are related via SLOCC, i.e., $\tilde{\sigma}^{ABC} = \Lambda_{\mathrm{SLOCC}}[\tilde{\rho}^{ABC}]$ with some stochastic tripartite LOCC operation Λ_{SLOCC}. We also prove certain cases when this map can be implemented with probability one.

Consider an LQICC protocol Λ_{LQICC} that maps ρ^{AB} into σ^{AB}. In the following, we assume that this protocol consists of n intermediate LQICC operations. If we introduce the states $\omega_0 = \rho$ and $\omega_n = \sigma$, then the total protocol can be written as $\omega_0^{AB} \to \omega_1^{AB} \to \cdots \to \omega_{n-1}^{AB} \to \omega_n^{AB}$. We further suppose that each step $\omega_k \to \omega_{k+1}$ is either a local quantum operation on Alice's side followed by classical communication of the outcome to Bob, or a local incoherent operation on Bob's side, followed by classical communication of the outcome to Alice. We will now see that for any such transformation $\omega_k^{AB} \to \omega_{k+1}^{AB}$ there exists a tripartite SLOCC protocol transforming $\tilde{\omega}_k^{ABC}$ to $\tilde{\omega}_{k+1}^{ABC}$.

First, suppose that the process $\omega_k^{AB} \to \omega_{k+1}^{AB}$ involves a local measurement of Alice and classical communication to Bob. Then, it is easy to see that the process $\tilde{\omega}_k^{ABC} \to \tilde{\omega}_{k+1}^{ABC}$ can be implemented deterministically, i.e., there exists a tripartite LOCC operation such $\tilde{\omega}_k^{ABC} \to \tilde{\omega}_{k+1}^{ABC}$. For this, the same local measurement has to be performed on the subsystem A of $\tilde{\omega}_k^{ABC}$, and the result is communicated to both parties B and C.

In the following we will consider the situation where the process $\omega_k^{AB} \to \omega_{k+1}^{AB}$ involves a local incoherent operation on Bob's side, followed by classical communication to Alice. We suppose that the state ω_k^{AB} has the form

$$\omega_k^{AB} = \sum_{i,j} O_{ij}^A \otimes |i\rangle\langle j|^B. \tag{A.34}$$

The incoherent operation performed by Bob can always be described by the following incoherent Kraus operators:

$$K_\alpha^B = \sum_i c_{\alpha,i} \left| f_\alpha(i) \right\rangle \langle i|^B , \tag{A.35}$$

where $c_{\alpha,i}$ are complex numbers, and the set of functions $f_\alpha(i)$ maps the set $\{i\}$ onto itself. If Bob obtains the outcome α, the corresponding post-measurement state is given by

$$\nu_\alpha^{AB} = \sum_{i,j} \frac{c_{\alpha,i} c_{\alpha,j}^*}{p_\alpha} O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \tag{A.36}$$

with probability

$$p_\alpha = \mathrm{Tr}\left[\sum_{i,j} c_{\alpha,i} c_{\alpha,j}^* O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \right]. \tag{A.37}$$

Correspondingly, the state $\tilde{\omega}_k^{ABC}$ has the form

$$\tilde{\omega}_k^{ABC} = \sum_{i,j} O_{ij}^A \otimes |ii\rangle \langle jj|^{BC} . \tag{A.38}$$

For showing the existence of a stochastic LOCC protocol transforming $\tilde{\omega}_k^{ABC}$ to $\tilde{\omega}_{k+1}^{ABC}$ it is enough to show that the state $\tilde{\omega}_k^{ABC}$ can be transformed into the state

$$\tilde{\nu}_\alpha^{ABC} = \sum_{i,j} \frac{c_{\alpha,i} c_{\alpha,j}^*}{p_\alpha} O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^C . \tag{A.39}$$

via stochastic LOCC operations with nonzero probability for all α. This protocol consists of the following steps.

1. In the first step, the incoherent measurement with Kraus operators $\{K_\alpha^B\}$ as given in Eq. (A.35) is performed on the party B of the total state $\tilde{\omega}_k^{ABC}$. If the outcome α is not possible in the LQICC protocol (i.e. if $p_\alpha = 0$), the protocol is aborted. Otherwise, with probability

$$q_\alpha = \mathrm{Tr}[K_\alpha^B \tilde{\omega}_k^{ABC} (K_\alpha^B)^\dagger] \tag{A.40}$$

(which is in general different from p_α) the outcome α is obtained and broadcast to the other parties A and C. The corresponding post-measurement state has the form

$$\tau_\alpha^{ABC} = \sum_{i,j} \frac{c_{\alpha,i} c_{\alpha,j}^*}{q_\alpha} O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \otimes |i\rangle \langle j|^C . \tag{A.41}$$

2. In the next step, Charlie introduces an ancilla system $\tilde{C}$ originally in the state $|0\rangle^{\tilde{C}}$ so that the total state is

$$\tau_\alpha^{ABC\tilde{C}} = \sum_{i,j} \frac{c_{\alpha,i} c_{\alpha,j}^*}{q_\alpha} O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \otimes |i\rangle \langle j|^C \otimes |0\rangle \langle 0|^{\tilde{C}} . \tag{A.42}$$

Depending on the outcome α Charlie then performs a local unitary rotation such that

$$U_\alpha \left(|i\rangle^C |0\rangle^{\tilde{C}} \right) = \left| f_\alpha(i) \right\rangle^C |i\rangle^{\tilde{C}} . \tag{A.43}$$

This takes $\tau_\alpha^{ABC\tilde{C}}$ to the state

$$\mu_\alpha^{ABC\tilde{C}} = \sum_{i,j} \frac{c_{\alpha,i} c_{\alpha,j}^*}{q_\alpha} O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^C \otimes |i\rangle \langle j|^{\tilde{C}} . \tag{A.44}$$

3. In the final step, Charlie measures $\tilde{C}$ in the generalized Hadamard basis: $\{|b_k\rangle = \frac{1}{\sqrt{d_B}} \sum_{j=0}^{d_B-1} e^{2\pi i k j/d_B} |j\rangle\}_{k=0}^{d_B-1}$. With some probability, outcome $|b_0\rangle$ is obtained, leading to the desired the final state $\tilde{\nu}_\alpha^{ABC}$ given in Eq. (A.39).

In the following we will show that the above procedure can always be implemented with nonzero probability. In particular, we will see that for any α with probability $p_\alpha > 0$ as described above, the probability to obtain the state $\tilde{\nu}_\alpha^{ABC}$ from the state $\tilde{\omega}_k^{ABC}$ via tripartite SLOCC is always nonzero.

To prove this, we will first show that $p_\alpha > 0$ implies $q_\alpha > 0$, where q_α was given in Eq. (A.40). This can be seen by contradiction, assuming that $q_\alpha = 0$. This implies the following:

$$\mathrm{Tr}\left[q_\alpha \tau_\alpha^{ABC} \mathbb{1}^{AB} \otimes |b_0\rangle \langle b_0| \right] = 0, \tag{A.45}$$

where the state $|b_0\rangle$ is given as $|b_0\rangle = \sum_{j=0}^{d_C-1} |j\rangle / \sqrt{d_C}$, and the particles B and C have the same dimension. This result together with Eq. (A.41) leads to the equality

$$\frac{1}{d_C} \mathrm{Tr}\left[\sum_{i,j} c_{\alpha,i} c_{\alpha,j}^* O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \right] = 0. \tag{A.46}$$

By comparing this with Eq. (A.37) we see that the left-hand side of this equality is equal to p_α / d_C, and thus $p_\alpha = 0$. This proves that $p_\alpha > 0$ implies $q_\alpha > 0$.

To complete the proof that the above procedure can always be accomplished with nonzero probability we note that in the measurement in the step 3 of the protocol the desired outcome appears with nonzero probability whenever $p_\alpha > 0$. This can be seen directly, by evaluating the corresponding probability:

$$\mathrm{Tr}\left[\mu_\alpha^{ABC\tilde{C}} \mathbb{1}^{ABC} \otimes |b_0\rangle \langle b_0| \right] = \mathrm{Tr}\left[\sum_{i,j} \frac{c_{\alpha,i} c_{\alpha,j}^*}{q_\alpha d_B} O_{ij}^A \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^B \otimes \left| f_\alpha(i) \right\rangle \left\langle f_\alpha(j) \right|^C \right]. \tag{A.47}$$

By comparing this expression with Eq. (A.37), we further find that

$$\mathrm{Tr}\left[\mu_\alpha^{ABC\tilde{C}} \mathbb{1}^{ABC} \otimes |b_0\rangle \langle b_0| \right] = \frac{p_\alpha}{q_\alpha d_B}. \tag{A.48}$$

Since we assume that $p_\alpha > 0$, this completes the proof that the stochastic LOCC procedure discussed above has always nonzero probability of success.

Finally, we note that for the certain types of incoherent operation Λ_{LQICC} the aforementioned transformation is deterministic. In particular, this is the case if the function f_α is reversible. Then there exists a unitary rotation for Charlie U_α^B such that

$$U_\alpha^C \left|i\right\rangle^C = \left|f_\alpha(i)\right\rangle^C . \tag{A.49}$$

Performing this rotation on the state in Eq. (A.41) generates the desired maximally correlated state $\tilde{\nu}_\alpha^{ABC}$, and steps 2 and 3 in the above protocol are omitted.

In summary, the transformation $\tilde{\rho}^{ABC} \rightarrow \tilde{\sigma}^{ABC}$ can always be achieved with some nonzero probability. If all the incoherent operations in $\Lambda_{\rm LQICC}$ have Kraus operators K_α with $f_\alpha(i)$ being reversible for every α, then the transformation can be accomplished with probability one.