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Abstract

We study the evolution of geometric quantum discord (GMQD) of a two qubits system coupled with two independent bosonic reservoirs. We consider sub-ohmic, ohmic and super-ohmic. A special attention is devoled to Dicke states and their superpositions.

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1 Introduction

Multipartite entanglement is one of the important areas in the field of quantum information that has many applications including quantum secret sharing. In this paper, we focus on the Dicke states [?], which are useful building blocks in realizing multipartite entanglement. The n-qubit weight w Dicke state, $|D_w^n\rangle$, is the equal superposition of all n-qubit states of weight w. We refer to [?, ?, ?, ?, ?, ?, ?] and the references therein for detailed discussion.

After the invention of quantum information, many experimental setups have been proposed and tested to verify some theoretical properties. Most of experiments have been focused on the test of multipartite entanglement such as EPR, GHZ, and W states. Since the result of experimental tests depends on the steps for preparing, processing, and measuring, all steps should be refined as much as possible. Among them, the first priority is to prepare the target state with very high fidelity and with efficiency. In this work, therefore, we also focus on the efficient way to prepare certain multipartite quantum state.

In line of GHZ and W states, we have the Dicke state, $|D_w^n\rangle$, which an equal superposition state of all n-qubit states of weight w. Actually, Dicke state is more general state than GHZ and W states since W state is $|D_1^n\rangle$ and GHZ state is the superposition of $|D_0^n\rangle$ and $|D_n^n\rangle$. Therefore, the preparation method for Dicke state can be utilized for other general case as well. At the same time, similar to the above reason, Dicke state can be utilized for many applications such as secret sharing [?] and quantum networking [?]. Related to this, some previous works have been done that focussed on the experimental ways to prepare six-qubit Dicke state [?, ?] with fidelity 0.654 ± 0.024 and 0.56 ± 0.02, respectively.

While the main focus from the viewpoint of experimental physics is to actually provide the implementation of specific Dicke states, our focus is from theoretical algorithmic angle and the only result presented in this direction appeared in [?]. In this work, we show how one can efficiently construct Dicke states by using the combinatorial properties of symmetric Boolean functions, two well-known quantum algorithms, and the generalized parity measurement. By efficient, we mean that the resource requirements in terms of quantum circuits and number of execution steps is poly(n) to obtain $|D_w^n\rangle$.

Let us consider n-qubit states in the computational basis $\{0,1\}^n$ that can be written in the form $\sum_{x\in\{0,1\}^n} a_x|x\rangle$, where $\sum_{x\in\{0,1\}^n} |a_x|^2 = 1$. Thus, x can also be interpreted as a binary string and the number of 1's in the string is called the (Hamming) weight of x and denoted as $wt(x)$. Based on this an arbitrary Dicke state can be expressed as follows:

$$|D_w^n\rangle = \sum_{x\in\{0,1\}^n, wt(x)=w} \frac{1}{\sqrt{\binom{n}{w}}}|x\rangle.$$

Let us also define a symmetric n-qubit state as

$$|S^n\rangle = \sum_{x\in\{0,1\}^n} a_{wt(x)}|x\rangle, \text{ where } \sum_{i=0}^{n} \binom{n}{i}|a_i|^2 = 1.$$

First, we show how one can prepare a symmetric n-qubit state with the property that $\binom{n}{w}|a_w|^2$ is $\Omega(\frac{1}{\sqrt{n}})$ by using Deutsch-Jozsa algorithm [?]. This requires certain novel combinatorial observations related to symmetric Boolean functions. Then the quantum state out of Deutsch-Jozsa algorithm is measured using the parity measurement technique [?] to obtain $|D^n_w\rangle$ with a probability $\Omega(\frac{1}{\sqrt{n}})$. Thus, $O(\sqrt{n})$ runs are sufficient to obtain the required Dicke state. Note that a direct approach to construct a symmetric state has been presented in [?] using biased Hadamard transform. While the order of probability to obtain Dicke state by ours and that of [?] are the same, enumeration results show that the exact probability values are better in our case than that of [?].

Further, motivated by the idea in [?], we improve our algorithm further with a modified Deutsch-Jozsa operator that involves the biased Hadamard transform. Since biased Hadamard transform also helps to generate the target symmetric state, the overall probability to obtain the Dicke state increases.

Finally, we can also apply the Grover operator [?] before the measurement. Since Grover algorithm amplifies the amplitude of target symmetric state, this helps to reduce the necessary number of steps into $O(\sqrt[4]{n})$.

2 Majorana Representation

The permutation symmetric subspace of n qubits is spanned by the Dicke states

$$|S(n,k)\rangle := \frac{1}{\sqrt{\binom{n}{k}}}\Big(\sum_{PERM}|\underbrace{00...0}_{n-k}\underbrace{11..1}_{k}\rangle\Big), \tag{1}$$

which can be understood as the symmetric states with k excitations. Thus any permutation symmetric state can be written as

$$|\psi\rangle = \sum a_k |S(n,k)\rangle. \tag{2}$$

Alternatively all symmetric states can be written in the Majorana representation [?]

$$|\psi\rangle = \frac{e^{i\alpha}}{\sqrt{K}} \sum_{PERM} |\eta_1\rangle|\eta_2\rangle...|\eta_n\rangle, \tag{3}$$

where the sum is over all permutations and K is a normalisation constant.

To find the Majorana representation (3) we consider the overlap with product state $|\phi\rangle^{\otimes n}$,$|\phi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$. It is clear by comparison to equation (3) that $|\phi\rangle$ orthoganol to the MP $|\eta_i\rangle$ will give zero overlap. This is exactly how we find the MPs. For simplicity we take a multiple of the overlap, sometimes called the *characteristic polynomial*, *Majorana polynomial*, *amplitude function* or *coherent state decomposition*

$$f(\psi) := \cos^{-n}\left(\frac{\theta}{2}\right)\langle\phi|^{\otimes n}|\psi\rangle = \sum_{k=0}^{n}\sqrt{\binom{n}{k}}a_k\alpha^k, \tag{4}$$

which is a complex polynomial in $\alpha := e^{-i\varphi} \tan\left(\frac{\theta}{2}\right)$. By the fundamental theorem of algebra this has unique zeros up to multiplication by some complex. Hence the zeros $\alpha_j = e^{-i\varphi_j} \tan\left(\frac{\theta_j}{2}\right)$ define the state $|\psi\rangle$ up to a global phase. The corresponding MPs are at position $\theta_j' = \theta_j + \pi$, $\varphi_j' = \varphi_j + \pi$.

Note that we can understand the state $|\phi\rangle^{\otimes n}$ as a kind of generalized coherent state [**?**, **?**], defined by the action of a group on some chosen fiducial state (so that certain properties apply such as overcompleteness). For our case the group is SU(2) as represented by $U^{\otimes n}$, with U a rotation through θ, φ and the fiducial state $|0\rangle^{\otimes n}$, that is

$$|\phi\rangle^{\otimes n} = U^{\otimes n}|0\rangle^{\otimes n}. \tag{5}$$

When viewing the symmetric subspace as one spin $S = n/2$ system, these are equivalent to spin coherent states [**?**, **?**]. In this sense the Majorana representation is a kind of condensed coherent state representation of states (since it is only concerned with the zeros of the coherent state decomposition (4)).

The Majorana representation then allows us to identify symmetries to show equality of the entanglement measures for many new sets of states. The complete set of all the possible subgroups of SO(3) are the continuous groups, orthogonal $O(2)$ and special orthogonal $SO(2)$, and discrete groups Cyclic C_m, Dihedral D_m, Tetrahedral T, Octehedral O and Isocahedral Y. One can then systematically go through all of these groups to find these special states, as done in [**?**] in the context of inert states. For the subgroup of arbitrary rotations about a fixed axis SO(2), we see that states with MPs only at either pole of the rotation axis satisfy our condition. If the rotations are around the Z-axis, these are the states

$$|S(n,k)\rangle := \frac{1}{\sqrt{\binom{n}{k}}}\Big(\sum_{PERM} |\underbrace{00...0}_{n-k}\underbrace{11..1}_{k}\rangle\Big),$$

also known as Dicke states, and we can see here pictorially the proof of equivalence for these states reported in [**?**]. Note that for even n and $k = n/2$, these states also satisfy our condition for the group $O(2)$ (arbitrary rotations around the Z-axis, and a flip on some axis in the $X-Y$ plane). In such cases we associate the state with the smallest subgroup. The cyclic group C_n has no truly invariant states - since if all points are moved together up and down the axis of rotation the symmetry is not lost. The dihedral group D_m (consisting of rotations through $2\pi/m$ and a flip on the axis of rotation) has m totally invariant states for each value m (see Fig. **??**). T, O and Y only have truly invariant states for certain n. For the tetrahedral group T truly invariant states are made up of tetrahedrons, their antipode tetrahedrons, and octagons with at most 2 MPs on any tetrahedron point and 3 MPs at any octahedron point, so that there are only truly invariant states for $n \leq 34$. For the octahhedral group O truly invariant states have MPs at the points of the cube and the octahedron with at most 3 and 2 MPs at each respectively, so that they only exist for $n \leq 34$. All truly invariant states of the Isocahedral group Y are made up of combinations of isocahedrons (with 12 vertices) and dodecahedrons (with 20 vertices) with at most 3 and 2 MPs at each respectively, hence they exist only for $n \leq 88$. For four qubits there are four entangled states satisfying the condition, $|T\rangle = 1/\sqrt{3}|S(4,0)\rangle + \sqrt{2/3}|S(4,3)\rangle$, $|GHZ_4\rangle$, $|S(4,2)\rangle$ and $|W_4\rangle = |S(4,1)\rangle$ as shown in Fig. **??**.

3 Symmetric multi-qubit systems

The multi-qubit symmetric states were shown relevant for different purposes in quantum information science [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]. In this paper, we shall mainly focus on an ensemble of n spin-1/2 prepared in even and odd spin coherent states.

3.1 Spin coherent as symmetric multi-qubit systems

We consider n identical qubits. Each qubit lives in a 2-dimensional Hilbert space $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$. The Hilbert space of the n-qubit system is given by n tensored copies of $\mathcal{H}$

$$\mathcal{H}_n := \mathcal{H}^{\otimes n}.$$

Among the multi-partite states in $\mathcal{H}_n$, multi-qubit states obeying exchange symmetry are of special interest from experimental as well as mathematical point of views. An arbitrary symmetric n-qubit state is commonly represented in either Majorana [**?**] or Dicke [**?**] representation. Any multi-qubit state, invariant under the exchange symmetry, is specified in the Majorana description by the state (up to a normalization factor)

$$|\psi_s\rangle = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} |\eta_{\sigma(1)}, \ldots, \eta_{\sigma(n)}\rangle, \tag{6}$$

where each single qubit state is $|\eta_i\rangle \equiv (1 + \eta_i \bar{\eta}_i)^{-\frac{1}{2}}(|0\rangle + \eta_i|1\rangle)$ $(i = 1, \ldots, n)$ and the sum is over the elements of the permutation group $\mathcal{S}_n$ of n objects. In Equation (6), the vector $|\eta_{\sigma(1)}, \ldots, \eta_{\sigma(n)}\rangle$ stands for the tensor product $|\eta_{\sigma(1)}\rangle \otimes \ldots \otimes |\eta_{\sigma(n)}\rangle$. The totally symmetric n-qubit states can be also formulated in Dike representation. The symmetric Dicke states with k excitations are defined by [**?**]

$$|n, k\rangle = \sqrt{\frac{k!(n-k)!}{n!}} \sum_{\sigma \in \mathcal{S}_n} |\underbrace{0, \ldots, 0}_{n-k}, \underbrace{1, \ldots, 1}_{k}\rangle, \tag{7}$$

which generate an orthonormal basis of the symmetric Hilbert subspace of dimension $(n+1)$. Therefore, permutation invariance, in symmetric multi-qubit states, implies a restriction to $n+1$ dimensional subspace from the entire 2^n dimensional Hilbert space. The Dicke states (7) constitute a special subset of the symmetric multi-qubit states (6) corresponding to the situation where the first k-qubit are such that $\eta_i = 0$ for $i = 0, 1, \ldots k$ and the remaining qubits are in the states $|\eta_i = 1\rangle$ with $i = k+1, \ldots, n$. Any symmetric state $|\psi_s\rangle$ (6) can be expanded in terms of Dicke states (7) as follows

$$|\psi_s\rangle = \frac{1}{n!} \sum_{k=0}^{n} c_k \; |n, k\rangle, \tag{8}$$

where the c_k ($k = 0, \dots, n$) stand for the complex expansion coefficients. In particular, when the qubit are all identical ($\eta_i = \eta$ for all qubits), it is simply verified that the coefficients c_k are given by

$$c_k = n! \sqrt{\frac{n!}{k!(n-k)!}} \frac{\eta^k}{(1+\eta\bar{\eta})^{\frac{n}{2}}} \tag{9}$$

and the symmetric multi-qubit states (6) write

$$|\psi_s\rangle := |n, \eta\rangle = (1+\eta\bar{\eta})^{-\frac{n}{2}} \sum_{k=0}^{n} \sqrt{\frac{n!}{k!(n-k)!}} \eta^k \; |n, k\rangle, \tag{10}$$

which are exactly the $j = \frac{n}{2}$-spin coherent states (for more details see for instance [**?**]). In particular, the state $|n, \eta\rangle$ can be identified for $n = 1$ with spin-$\frac{1}{2}$ coherent state with $|0\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|1\rangle \equiv |\frac{1}{2}, +\frac{1}{2}\rangle$)

3.2 Multi-qubit "Shroödinger cat" states

The prototypical multi-qubit "Shrödinger cat" states, we consider in this work, are defined as a balanced superpositions of the n-qubit states $|n, \eta\rangle$ and $|n, -\eta\rangle$ given by (10). They write

$$|\eta, n, m\rangle = \mathcal{N}(|n, \eta\rangle + e^{im\pi}|n, -\eta\rangle) \tag{11}$$

where

$$|n, \pm\eta\rangle = |\pm\eta\rangle \otimes |\pm\eta\rangle \cdots \otimes |\pm\eta\rangle,$$

and the integer $m \in \mathbb{Z}$ takes the values $m = 0$ (mod 2) and $m = 1$ (mod 2). The normalization factor $\mathcal{N}$ is

$$\mathcal{N} = \left[2 + 2p^n \cos m\pi\right]^{-1/2}$$

where p denotes the overlap between the states $|\eta\rangle$ and $|-\eta\rangle$. It is given by

$$p = \langle\eta| - \eta\rangle = \frac{1 - \bar{\eta}\eta}{1 + \bar{\eta}\eta}. \tag{12}$$

Experimental creation of cat states comprising multiple particles was reported in the literature [**?**, **?**]. Due to their experimental implementation, "Shrödinger cat" states are expected to be an useful resource for quantum computing as well as quantum communications. Also, in view of their mathematical elegance, multi-qubit states obeying exchange symmetry offer drastic simplification in investigating various aspects of quantum correlations in particular the geometric measure of quantum discord as we shall discuss in the present work. Furthermore, the multi-qubit symmetric states (11) include Greenberger-Horne-Zeilinger(GHZ) [**?**], W [**?**] and Dicke states [**?**]. The multi-qubits states $|n, \eta, 0\rangle$ ($m = 0$ mod 2) and $|n, \eta, 1\rangle$ ($m = 1$ mod 2) behave like a multipartite state of Greenberger-Horne-Zeilinger (GHZ) type [**?**] in the limiting case $p \to 0$. Indeed, the states $|\eta\rangle$ and $|-\eta\rangle$ approach orthogonality and an orthogonal basis can be defined such that $|\mathbf{0}\rangle \equiv |\eta\rangle$ and $|\mathbf{1}\rangle \equiv |-\eta\rangle$. Thus, the state $|n, \eta, m\rangle$ becomes of GHZ-type:

$$|\eta, n, m\rangle \sim |\text{GHZ}\rangle_n = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \cdots \otimes |\mathbf{0}\rangle + e^{im\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes \cdots \otimes |\mathbf{1}\rangle). \tag{13}$$

Also, in the special situation where the overlap p tends to unity ($p \to 1$ or $\eta \to 0$), the state $|\eta, n, m = 0 \pmod 2\rangle$ (11) reduces to ground state of a collection of n qubits

$$|0, n, 0 \pmod 2\rangle \sim |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle, \tag{14}$$

and it is simple to check that the state $|\eta, 0, 1 \pmod 2\rangle$ becomes a multipartite state of W type [?]

$$|0, n, 1 \pmod 2\rangle \sim |\mathrm{W}\rangle_n = \frac{1}{\sqrt{n}}(|1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes \ldots \otimes |0\rangle + \cdots + |0\rangle \otimes |0\rangle \otimes \cdots \otimes |1\rangle) \,. \tag{15}$$

It is clear that the Shrödinger cat states $|\eta, n, m = 0 \pmod 2\rangle$ include the GHZ_n states ($p \to 0$). In other hand, the states $|\eta, n, m = 1 \pmod 2\rangle$, constitute an interpolation between two special classes of multi-qubits states: $|\mathrm{GHZ}\rangle_n$ type corresponding to $p \to 0$ and states of $|\mathrm{W}\rangle_n$ type obtained in the special case where $p \to 1$.

4 Conclusion

In this paper, in order to investigate the pairwise quantum correlations, we have studied the dynamics of GMQD The model under consideration consists of two qubits coupled with two independent bosonic reservoirs described by Ohmic-like spectral densities. The Hamiltonian of the model is described as Eq (**??**), and the two-qubit system is initially in the X states decoupled from the environments, described by Eq (**??**). We examine the evolution of GMQD for the three types of reservoirs, sub-ohmic, ohmic, and super-ohmic particularly the Dicke states and their superpositions. D_G is a monotonic decreasing function of time t or a poiecewise monotonic decreasing function with one turning point before becoming frozen phenomenon.

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