Frozen Quantum Coherence

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We analyse under which dynamical conditions the coherence of an open quantum system is totally unaffected by noise. For a single qubit, specific measures of coherence are found to freeze under different conditions, with no general agreement between them. Conversely, for an N-qubit system with even N, we identify universal conditions in terms of initial states and local incoherent channels such that all bona fide distance-based coherence monotones are left invariant during the entire evolution. This finding also provides an insightful physical interpretation for the freezing phenomenon of quantum correlations beyond entanglement. We further obtain analytical results for distance-based measures of coherence in two-qubit states with maximally mixed marginals.

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Introduction. The coherent superposition of states stands as one of the characteristic features that mark the departure of quantum mechanics from the classical realm, if not *the* most essential one [1]. Quantum coherence constitutes a powerful resource for quantum metrology [2, 3] and entanglement creation [4, 5], and is at the root of a number of intriguing phenomena of wide-ranging impact in quantum optics [6–9], quantum information [10], solid state physics [11, 12], and thermodynamics [13–18]. In recent years, research on the presence and functional role of quantum coherence in biological systems has also attracted a considerable interest [19–35].

Despite the fundamental importance of quantum coherence, only very recently have relevant first steps been achieved towards developing a rigorous theory of coherence as a physical resource [36–38], and necessary constraints have been put forward to assess valid quantifiers of coherence [36, 39]. A number of coherence measures have been proposed and investigated, such as the l_1-norm and relative entropy of coherence [36], and the skew information [40, 41]. Attempts to quantify coherence via a distance-based approach, which has been fruitfully adopted for entanglement and other correlations [42–52], have revealed some subtleties [53].

A lesson learned from natural sciences is that coherence-based effects can flourish and persist at significant timescales under suitable exposure to decohering environments. Recent evidence suggests that a fruitful interplay between long-lived quantum coherence and tailored noise may be in fact crucial to enhance certain biological processes, such as light harvesting [27, 28, 30, 31]. This surprising cooperation between traditionally competing phenomena provides an inspiration to explore other physical contexts, such as quantum information science, in order to seek for general conditions under which coherence can be sustained in the presence of typical sources of noise [54, 55]. Progress on this fundamental question can lead to a more efficient exploitation of coherence to empower the performance of real-world quantum technologies.

In this Letter we investigate the dynamics of quantum coherence in open quantum systems under paradigmatic incoherent noisy channels. While coherence is generally nonincreasing under any incoherent channel [36], our goal is to identify initial states and dynamical conditions, here labelled *freezing conditions*, such that coherence will remain exactly constant (frozen) during the whole evolution (see Fig. 1).

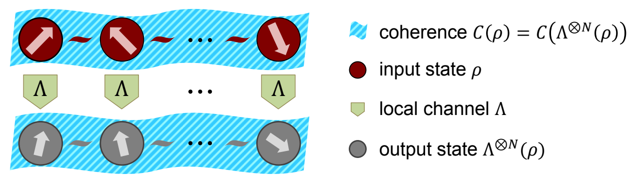

FIG. 1: (color online) Frozen quantum coherence for an N-qubit system subject to incoherent noisy channels Λ acting on each qubit.

For a single qubit subject to a Markovian bit flip, bit-phase flip, phase flip, depolarising, amplitude damping, or phase damping channel [10], we study the evolution of the l_1-norm and relative entropy of coherence [36] with respect to the computational basis. We show that no nontrivial condition exists such that both measures are simultaneously frozen. We then turn our attention to two-qubit systems, for which we remarkably identify a set of initial states such that *all* bona fide distance-based measures of coherence are frozen forever when each qubit is independently experiencing a nondissipative flip channel. These results are extended to N-qubit systems with any even N, for which suitable conditions supporting the freezing of all distance-based measures of coherence are provided. Such a *universal* freezing of quantum coherence within the geometric approach is intimately related to the freezing of distance-based quantum correlations beyond entanglement [50, 52, 56–58], thus shedding light on the latter from a physical perspective. Finally, some analytical results for the l_1-norm of coherence are obtained, and its freezing conditions in general one- and two-qubit states are identified.

Incoherent states and channels. Quantum coherence is conventionally associated with the capability of a quantum state to exhibit quantum interference phenomena [9]. Coherence effects are usually ascribed to the off-diagonal elements of a density matrix with respect to a particular reference basis, whose choice is dictated by the physical scenario under consideration [59]. Here, for an N-qubit system associated to a Hilbert space $\mathbb{C}^{2^N}$, we fix the computational basis $\{|0\rangle, |1\rangle\}^{\otimes N}$ as the reference basis, and we define incoherent states as those whose density matrix δ is diagonal in such a basis,

$$\delta = \sum\nolimits_{i_1,\ldots,i_N=0}^{1} d_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle\langle i_1,\ldots,i_N| \,. \tag{1}$$

Markovian dynamics of an open quantum system is described by a completely positive trace-preserving (CPTP) map Λ, i.e. a quantum channel, whose action on the state ρ of the system can be characterised by a set of Kraus operators $\{K_j\}$ such that $\Lambda(\rho) = \sum_j K_j \rho K_j^\dagger$, where $\sum_j K_j^\dagger K_j = \mathbb{I}$. Incoherent quantum channels (ICPTP maps) constitute a subset of quantum channels that satisfy the additional constraint $K_j \mathcal{I} K_j^\dagger \subset \mathcal{I}$ for all j, where $\mathcal{I}$ is the set of incoherent states [36]. This implies that ICPTP maps transform incoherent states into incoherent states, and no creation of coherence would be witnessed even if an observer had access to individual outcomes.

We will consider paradigmatic instances of incoherent channels which embody typical noise sources in quantum information processing [10, 36], and whose action on a single qubit is described as follows, in terms of a parameter $q \in [0,1]$ which encodes the strength of the noise. The bit flip, bit-phase flip and phase flip channels are represented in Kraus form by

$$K_0^{F_k} = \sqrt{1-q/2}\,\mathbb{I},\; K_{i,j\neq k}^{F_k} = 0,\; K_k^{F_k} = \sqrt{q/2}\,\sigma_k, \tag{2}$$

with $k=1$, $k=2$ and $k=3$, respectively, and σ_j being the j-th Pauli matrix. The depolarising channel is represented by $K_0^D = \sqrt{1-3q/4}\mathbb{I}$, $K_j^D = \sqrt{q/4}\sigma_j$, with $j \in \{1,2,3\}$. Finally, the amplitude damping channel is represented by $K_0^A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$, $K_1^A = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}$, and the phase damping channel by $K_0^P = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$, $K_1^P = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{q} \end{pmatrix}$.

The action of N independent and identical local noisy channels (of a given type, say labelled by $\Xi = \{F_k, D, A, P\}$) on each qubit of an N-qubit system, as depicted in Fig. 1, maps the system state ρ into the evolved state

$$\Lambda_q^{\Xi\otimes N}(\rho) = \sum_{j_1,\cdots,j_N} \left(K_{j_1}^{\Xi} \otimes \cdots \otimes K_{j_N}^{\Xi}\right) \rho \left(K_{j_1}^{\Xi\,\dagger} \otimes \cdots \otimes K_{j_N}^{\Xi\,\dagger}\right). \tag{3}$$

Coherence monotones. Baumgratz *et al.* [36] have formulated a set of physical requirements which should be satisfied by any valid measure of quantum coherence C, namely:

- *C1.* $C(\rho) \geq 0$ for all states ρ, with $C(\delta) = 0$ for all incoherent states $\delta \in \mathcal{I}$;
- *C2a.* Contractivity under incoherent channels Λ_{ICPTP}, $C(\rho) \geq C(\Lambda_{\mathrm{ICPTP}}(\rho))$;
- *C2b.* Contractivity under selective measurements on average, $C(\rho) \geq \sum_j p_j C(\rho_j)$, where $\rho_j = K_j \rho K_j^\dagger / p_j$ and $p_j = \mathrm{Tr}(K_j \rho K_j^\dagger)$, for any $\{K_j\}$ such that $\sum_j K_j^\dagger K_j = \mathbb{I}$ and $K_j \mathcal{I} K_j \subset \mathcal{I}$ for all j;
- *C3.* Convexity, $C(q\rho + (1-q)\tau) \leq qC(\rho) + (1-q)C(\tau)$ for any states ρ and τ and $q \in [0,1]$.

We now recall known measures of coherence. The l_1-norm quantifies coherence in an intuitive way, via the off-diagonal elements of a density matrix ρ in the reference basis [36],

$$C_{l_1}(\rho) = \sum\nolimits_{i\neq j} \left|\rho_{ij}\right| . \tag{4}$$

Alternatively, one can quantify coherence by means of a geometric approach. Given a distance D, a generic distance-based measure of coherence is defined as

$$C_D(\rho) = \min_{\delta\in\mathcal{I}} D(\rho,\delta) = D(\rho,\delta_\rho)\,, \tag{5}$$

where δ_ρ is one of the closest incoherent states to ρ with respect to D. We refer to bona fide distances D as those which satisfy natural properties [10] of contractivity under quantum channels, i.e. $D(\Lambda(\rho),\Lambda(\tau)) \leq D(\rho,\tau)$ for any states ρ,τ and CPTP map Λ, and joint convexity, i.e. $D(q\rho+(1-q)\varpi, q\tau+(1-q)\varsigma) \leq qD(\rho,\tau) + (1-q)D(\varpi,\varsigma)$ for any states $\rho,\varpi,\tau,\varsigma$ and $q \in [0,1]$. We then refer to bona fide distance-based measures of coherence C_D as those defined by Eq. (5) using a bona fide distance D: all such measures will satisfy requirements *C1*, *C2a*, and *C3* [36]. Additional contractivity requirements are needed for a distance D in order for the corresponding C_D to obey *C2b* as well [60]. For instance, while the fidelity-based geometric measure of coherence has been recently proven to be a full coherence monotone [5], a related coherence quantifier defined via the squared Bures distance (which is contractive and jointly convex) is known not to satisfy *C2b* [53].

All our subsequent findings will apply to bona fide distance-based coherence measures C_D, which clearly include coherence monotones obeying all the resource-theory requirements recalled earlier. An example of a distance-based coherence monotone is the relative entropy of coherence [36], given by

$$C_{RE}(\rho) = \mathcal{S}(\rho_{\mathrm{diag}}) - \mathcal{S}(\rho) \tag{6}$$

for any state ρ, where ρ_{diag} is the matrix containing only the leading diagonal elements of ρ in the reference basis, and $\mathcal{S}(\rho) = -\mathrm{Tr}(\rho\log\rho)$ is the von Neumann entropy.

We can also define the trace distance of coherence C_{Tr} as in Eq. (5) using the bona fide trace distance $D_{\mathrm{Tr}}(\rho,\tau) = \frac{1}{2}\mathrm{Tr}|\rho-\tau|$. For one-qubit states ρ, the trace distance of coherence equals (half) the l_1-norm of coherence [48, 53], but this equivalence is not valid for higher dimensional systems, and it is still unknown whether C_{Tr} obeys requirement *C2b* in general.

Frozen coherence: one qubit. We now analyse conditions such that the l_1-norm and relative entropy of coherence are invariant during the evolution of a single qubit (initially in a state ρ) under any of the noisy channels Λ_q^Ξ described above. This is done by imposing a vanishing differential of the measures on the evolved state, $\partial_q C(\Lambda_q^\Xi(\rho)) = 0 \;\; \forall q \in [0,1]$, with respect to the noise parameter q, which can also be interpreted as a dimensionless time [61]. We find that only the bit and bit-phase flip channels allow for nonzero frozen coherence (in the computational basis), while all the other considered incoherent channels leave coherence invariant only trivially when the initial state is already incoherent. We can then ask whether nontrivial common freezing conditions for C_{l_1} and C_{RE} exist.

Writing a single-qubit state in general as $\rho = \frac{1}{2}(\mathbb{I} + \sum_j n_j \sigma_j)$ in terms of its Bloch vector $\vec{n} = \{n_1, n_2, n_3\}$, the bit flip channel $\Lambda_q^{F_1}$ maps an initial Bloch vector $\vec{n}(0)$ to an evolved one $\vec{n}(q) = \{n_1(0), (1-q)n_2(0), (1-q)n_3(0)\}$. As the l_1-norm of coherence is independent of n_3, while n_1 is unaffected by the channel, we get that necessary and sufficient freezing conditions for C_{l_1} under a single-qubit bit flip channel amount to $n_2(0) = 0$ in the initial state. Similar conclusions apply to the bit-phase flip channel $\Lambda_q^{F_2}$ by swapping the roles of n_1 and n_2.

Conversely, the relative entropy of coherence is also dependent on n_3. By analysing the q-derivative of C_{RE}, we see that such a measure is frozen through the bit flip channel only when either $n_1(0) = 0$ and $n_2(0) = 0$ (trivial because the initial state is incoherent) or $n_2(0) = 0$ and $n_3(0) = 0$ (trivial because the initial state is invariant under the channel). Therefore, there is no nontrivial freezing of the relative entropy of coherence under the bit flip or bit-phase flip channel either.

We conclude that, although the l_1-norm of coherence can be frozen for specific initial states under flip channels, nontrivial universal freezing of coherence is impossible for the dynamics of a single qubit under paradigmatic incoherent maps.

Frozen coherence: two qubits. This is not true anymore when considering more than one qubit. We will now show that any bona fide distance-based measure of quantum coherence manifests freezing forever in the case of two qubits A and B undergoing local identical bit flip channels [62] and starting from the initial conditions specified as follows. We consider two-qubit states with maximally mixed marginals (M_2^3 states), also known as Bell-diagonal states [63], which are identified by a triple $\vec{c} = \{c_1, c_2, c_3\}$ in their Bloch representation

$$\rho = \frac{1}{4}\left(\mathbb{I}^A \otimes \mathbb{I}^B + \sum\nolimits_{j=1}^{3} c_j \sigma_j^A \otimes \sigma_j^B\right). \tag{7}$$

Local bit flip channels on each qubit map initial M_2^3 states with $\vec{c}(0) = \{c_1(0), c_2(0), c_3(0)\}$ to M_2^3 states with $\vec{c}(q) = \{c_1(0), (1-q)^2 c_2(0), (1-q)^2 c_3(0)\}$. Then, the subset of M_2^3 states supporting frozen coherence for all bona fide distance-based measures is given by the initial condition [50, 52, 57],

$$c_2(0) = -c_1(0)c_3(0)\,. \tag{8}$$

To establish this claim, we first enunciate two auxiliary results, which simplify the evaluation of distance-based coherence monotones (5) for the relevant class of M_2^3 states.

Lemma 1. According to any contractive and convex distance D, one of the closest incoherent states δ_ρ to a M_2^3 state ρ is always a M_2^3 incoherent state, i.e. one of the form

$$\delta_\rho = \tfrac{1}{4}\left(\mathbb{I}^A \otimes \mathbb{I}^B + s\sigma_3^A \otimes \sigma_3^B\right), \text{ for some } s \in [-1, 1]\,. \tag{9}$$

Lemma 2. According to any contractive and convex distance D, one of the closest incoherent states δ_ρ to a M_2^3 state ρ with triple $\{c_1, -c_1c_3, c_3\}$ is the M_2^3 state δ_ρ with triple $\{0, 0, c_3\}$.

It then follows that any bona fide distance-based measure of coherence C_D for the M_2^3 states $\rho(q)$, evolving from the initial conditions (8) under local bit flip channels, is given by

$$\begin{aligned} C_D(\rho(q)) \;&=\; D\Big(\{c_1(0), -(1-q)^2 c_1(0)c_3(0), (1-q)^2 c_3(0)\}, \\ &\qquad \{0, 0, (1-q)^2 c_3(0)\}\Big) \;=\; C_D(\rho(0))\,, \end{aligned}$$

which is frozen for any $q \in [0, 1]$, or equivalently frozen forever for any t [61]. The two Lemmas and the main implication on frozen coherence can be rigorously proven by invoking and adapting recent results on the dynamics of quantum correlations for M_2^3 states, reported in [52]. A comprehensive proof is provided in the Supplemental Material [64]. This finding shows that, in contrast to the one-qubit case, universal freezing of quantum coherence—measured within a bona fide geometric approach—can in fact occur in two-qubit systems exposed to conventional local decohering dynamics.

Coming back now to the two specific coherence monotones analysed here [36], we know that the relative entropy of coherence C_{RE} is a bona fide distance-based measure, hence it manifests freezing in the conditions of Eq. (8). Interestingly, we will now show that the l_1-norm of coherence C_{l_1} coincides with (twice) the trace distance of coherence C_{Tr} for any M_2^3 state, which implies that C_{l_1} also freezes in the same dynamical conditions. To this aim we need to show that, with respect to the trace distance D_{Tr}, one of the closest incoherent states δ_ρ to a M_2^3 state ρ is always its diagonal part ρ_{diag}. The trace distance between a M_2^3 state ρ with $\{c_1, c_2, c_3\}$ and one of its closest incoherent states δ_ρ, which is itself a M_2^3 state of the form (A.36) according to Lemma 1, is given by $D_{\mathrm{Tr}}(\rho, \delta_\rho) = \frac{1}{4}(|s + c_1 - c_2 - c_3| + |s - c_1 + c_2 - c_3| + |s + c_1 + c_2 - c_3| + |-s + c_1 + c_2 + c_3|)$. It is immediate to see that the minimum over δ_ρ is attained by $s = c_3$, i.e., by $\delta_\rho = \rho_{\mathrm{diag}}$ as claimed. Notice, however, that the equivalence between C_{l_1} and C_{Tr} does not extend to general two-qubit states, as can be confirmed numerically.

Similarly to the single-qubit case, we can derive a larger set of necessary and sufficient freezing conditions valid specifically for the l_1-norm of coherence. Every two-qubit state ρ can be transformed, by local unitaries, into a standard form [65] with Bloch representation $\rho = \frac{1}{4}\Big(\mathbb{I}^A \otimes \mathbb{I}^B + \sum_{j=1}^3 x_j \sigma_j^A \otimes \mathbb{I}^B + \sum_{j=1}^3 y_j \mathbb{I}^A \otimes \sigma_j^B + \sum_{j=1}^3 T_{jj} \sigma_j^A \otimes \sigma_j^B\Big)$. We have then that initial states of this form, with x_1, y_1, x_3, y_3, T_{33} arbitrary, $x_2 = y_2 = 0$, and $T_{22} = uT_{11}$ with $u \in [-1, 1]$, manifest frozen coherence as measured by C_{l_1} under local bit flip channels; however, the same does not hold for C_{RE} in general.

Frozen coherence: N qubits. Our main finding can be readily generalised to a system of N qubits with any even N. We define N-qubit states with maximally mixed marginals (M_N^3 states) [58, 66] as those with density matrix of the form $\rho = \frac{1}{2^N}\left(\mathbb{I}^{\otimes N} + \sum_{j=1}^3 c_j \sigma_j^{\otimes N}\right)$, still specified by the triple $\{c_1, c_2, c_3\}$ as in the $N = 2$ case. We have then that, when the system is evolving according to identical and independent local bit flip channels acting on each qubit as in Eq. (3) with $\Xi = F_1$, the quantum coherence of the system is universally frozen according to any bona fide distance-based measure if the N qubits are initialised in a M_N^3 state respecting the freezing condition

$$c_2(0) = (-1)^{N/2} c_1(0) c_3(0), \tag{10}$$

which generalises (8). This is the most general result of the present Letter [62], and its full proof is provided in the Supplemental Material [64]. We observe that, by virtue of the formal equivalence between a system of N qubits and a single qudit with dimension $d = 2^N$, our results can also be interpreted as providing universal freezing conditions for all bona fide distance-based measures of coherence in a single 2^N-dimensional system with any even N. Naturally, one may expect larger sets of freezing conditions to exist for specific coherence monotones such as the l_1-norm, like in the $N = 2$ case; their characterisation is outside the scope of this Letter.

We further note that no universal freezing of coherence is instead possible for M_N^3 states with odd N, whose dynamical properties are totally analogous to those of one-qubit states.

Coherence versus quantum correlations. The freezing conditions established here for coherence have been in fact identified in previous literature [50, 52, 56–58], as various measures of so-called discord-type quantum correlations were shown to freeze under the same dynamical conditions up to a threshold time $t^\star$, defined in our notation [61] by the largest value of q such that $|c_3(q)| \geq |c_1(q)|$, for M_N^3 states evolving under local bit flip channels. Focusing on the two-qubit case for clarity, we note that for M_2^3 states with $|c_3| \geq |c_1|$, and for any bona fide distance D, the distance-based measure of coherence C_D, defined by Eq. (5) and evaluated in Eq. (10), *coincides* with the corresponding distance-based measure of discord-type quantum correlations Q_D, formalised e.g. in Ref. [52]. Hence, the freezing of coherence might provide a deeper insight into the peculiar phenomenon of frozen quantum correlations under local flip channels (see also [67]), as the latter just reduce to coherence for $t \leq t^\star$ under the conditions we identified.

More generally, measures of discord-type correlations [46, 68, 69] may be recast as suitable measures of coherence in bipartite systems, minimised over the reference basis, with minimisation restricted to local product bases. For instance, the minimum l_1-norm of coherence [36] yields the negativity of quantumness [48, 70, 71], the minimum relative entropy of coherence [36] yields the relative entropy of discord [45, 70, 72, 73], and the minimum skew information [40] yields the local quantum uncertainty [74]. Our result suggest therefore that the computational basis is the product basis which minimises coherence (according to suitable bona fide measures) for particular M_2^3 states undergoing local bit flip noise Λ^{F_1} up to $t \leq t^\star$, while coherence is afterwards minimised in the eigenbasis of σ_1, which is the pointer basis towards which the system eventually converges due to the local decoherence [75]; similar conclusions can be drawn for the other k-flip channels [62].

We finally remark that, unlike more general discord-type correlations, entanglement [44] plays no special role in the freezing phenomenon analysed in this Letter, as the latter can also happen for states that remain separable during the whole evolution, e.g. the M_2^3 states with initial triple $\{\frac{1}{4}, -\frac{1}{16}, \frac{1}{4}\}$.

Conclusions. We have determined exact conditions such that any bona fide distance-based measure of quantum coherence [36] is dynamically frozen: this occurs for an even number of qubits, initialised in a particular class of states with maximally mixed marginals, and undergoing local independent and identical nondissipative flip channels (Fig. 1). We have also shown that there is no general agreement on freezing conditions between specific coherence monotones when considering either the one-qubit case or more general N-qubit initial states. This highlights the prominent role played by the aforementioned universal freezing conditions in ensuring a durable physical exploitation of coherence, regardless of how it is quantified, for applications such as quantum metrology [2] and nanoscale thermodynamics [17, 18]. It will be interesting to explore practical realisations of such dynamical conditions [75–80].

Complex systems are inevitably subject to noise, hence it is natural and technologically crucial to question under what conditions the quantum resources that we can extract from them are not deteriorated during open evolutions [81]. In addressing this problem by focusing on coherence, we have also revealed an intrinsic physical explanation for the freezing of discord-type correlations [52], by exposing and exploiting the intimate link between these two nonclassical signatures. Providing unified quantitative resource-theory frameworks for coherence, entanglement, and other quantum correlations is certainly a task worthy of further investigation [5].

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Appendix: Supplemental Material

Consider the N-qubit states with the following matrix representation in the computational basis:

$$\rho = \frac{1}{2^N}\left(\mathbb{I}^{\otimes N} + \sum_{i=1}^{3} c_i \sigma_i^{\otimes N}\right), \tag{A.11}$$

where $\mathbb{I}$ is the 2×2 identity matrix, σ_i is the i-th Pauli matrix and $c_i = \mathrm{Tr}\left[\rho\sigma_i^{\otimes N}\right] \in [-1,1]$. These states will be referred to as M_N^3 states, as they have maximally mixed marginals (by tracing out any $K < N$ qubits), and will be uniquely identified by the triple $\{c_1, c_2, c_3\}$.

In this appendix we will show that, for an even number N of qubits, all bona fide distance-based measures of quantum coherence will exhibit the freezing phenomenon when each qubit is subject to local independent bit flip noise, for an initial M_N^3 state specified by $\{c_1, (-1)^{N/2}c_1c_3, c_3\}$.

The evolution of an N-qubit state ρ under local independent identical k-flip channels, where the index $k \in \{1,2,3\}$ respectively identifies the bit flip ($k=1$), bit-phase flip ($k=2$), and phase flip ($k=3$) channel, can be characterised in the operator-sum representation by the map

$$\Lambda_q^{F_k\otimes N}(\rho) = \sum_{j_1,j_2,\cdots,j_N} K_{j_1}^{F_k}\otimes K_{j_2}^{F_k}\otimes\cdots\otimes K_{j_N}^{F_k}\rho K_{j_1}^{F_k\dagger}\otimes K_{j_2}^{F_k\dagger}\otimes\cdots\otimes K_{j_N}^{F_k\dagger} \tag{A.12}$$

where the single-qubit Kraus operators $K_j^{F_k}$ are reported in the main text in terms of the strength of the noise $q \in [0,1]$, which in dynamical terms can be expressed as $q(t) = 1 - \exp(-\gamma t)$ with t representing time and γ being the decoherence rate. From Eqs. (A.11) and (A.12), one can easily see that N non-interacting qubits initially in a M_N^3 state, undergoing local identical flip channels, evolve preserving the M_N^3 structure during the entire dynamics (i.e, for all $q \in [0,1]$, or equivalently for all $t \geq 0$). More precisely, the triple $\{c_1(q), c_2(q), c_3(q)\}$ characterising the M_N^3 evolved state $\rho(q)$ can be written as follows

$$c_{i,j\neq k}(q) = (1-q)^N c_{i,j\neq k}(0), \quad c_k(q) = c_k(0), \tag{A.13}$$

where $\{c_1(0), c_2(0), c_3(0)\}$ is the triple characterising the initial M_N^3 state ρ.

We start by showing that, for even N, the eigenvectors and eigenvalues of an arbitrary M_N^3 state ρ are given by, respectively

$$\begin{aligned}
|\beta_1^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|000\ldots000\rangle \pm |111\ldots111\rangle\right), \qquad \text{(A.14)}\\
|\beta_2^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|000\ldots001\rangle \pm |111\ldots110\rangle\right),\\
|\beta_3^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|000\ldots010\rangle \pm |111\ldots101\rangle\right),\\
|\beta_4^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|000\ldots011\rangle \pm |111\ldots100\rangle\right),\\
&\ldots\\
|\beta_{2^{N-1}-1}^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|011\ldots110\rangle \pm |100\ldots001\rangle\right),\\
|\beta_{2^{N-1}}^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|011\ldots111\rangle \pm |100\ldots000\rangle\right),
\end{aligned}$$

and

$$\lambda_p^\pm = \frac{1}{2^N}\left[1 \pm c_1 \pm (-1)^{N/2}(-1)^p c_2 + (-1)^p c_3\right], \tag{A.15}$$

where p is the parity of $|\beta_i^\pm\rangle$ with respect to the parity operator along the z-axis $\Pi_3 \equiv \sigma_3^{\otimes N}$, i.e.

$$\Pi_3|\beta_i^\pm\rangle = (-1)^p|\beta_i^\pm\rangle. \tag{A.16}$$

It will suffice to prove the following equation:

$$\rho|\beta_i^\pm\rangle = \lambda_p^\pm|\beta_i^\pm\rangle, \tag{A.17}$$

for any $i \in \{1,\cdots,2^{N-1}\}$. In fact, by writing a generic state $|\beta_i^\pm\rangle$ as follows

$$|\beta_i^\pm\rangle = \frac{1}{\sqrt{2}}\left(|p, N-p\rangle \pm |N-p, p\rangle\right), \tag{A.18}$$

where $|n_0, n_1\rangle$ denotes any element of the N-qubit computational basis whose number of 0's (1's) is equal to n_0 (n_1), one can easily see that

$$\begin{aligned}
\sigma_1^{\otimes N}|\beta_i^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|N-p, p\rangle \pm |p, N-p\rangle\right),\\
\sigma_2^{\otimes N}|\beta_i^\pm\rangle &= \frac{1}{\sqrt{2}}(-1)^{N/2}\left[(-1)^p|N-p,p\rangle \pm (-1)^{N-p}|p, N-p\rangle\right],\\
\sigma_3^{\otimes N}|\beta_i^\pm\rangle &= \frac{1}{\sqrt{2}}\left[(-1)^{N-p}|p, N-p\rangle \pm (-1)^p|N-p, p\rangle\right].
\end{aligned}$$

Eventually, by using the above three equations, Eq. (A.11) and the fact that N is even, so that $(-1)^{N-p} = (-1)^p$, one can easily verify that $\rho|\beta_i^\pm\rangle$ is equal to $\lambda_p^\pm|\beta_i^\pm\rangle$, i.e. that Eq. (A.17) holds.

Now we are ready to show the three essential pieces which will lead us to prove the main result on the universal freezing phenomenon of bona fide distance-based measures of quantum coherence in the N-qubit setting (with even N).

Lemma A.1. *For all even N, any contractive distance satisfies the following translational invariance properties within the space of N-qubit M_N^3 states:*

$$D(\{c_1,(-1)^{N/2}c_1c_3,c_3\},\{c_1,0,0\}) = D(\{0,0,c_3\},\{0,0,0\}) \tag{A.19}$$

and

$$D(\{c_1,(-1)^{N/2}c_1c_3,c_3\},\{0,0,c_3\}) = D(\{c_1,0,0\},\{0,0,0\}) \tag{A.20}$$

for all c_1 and c_3, where $\{c_1,(-1)^{N/2}c_1c_3,c_3\}$ denotes a M_N^3 state in Eq. (A.11) with $c_2=(-1)^{N/2}c_1c_3$.

Proof. Let us start by proving Eq. (A.19). First of all, by considering the channel $\Lambda_1^{F_3\otimes N}$ representing the local independent phase flip noise expressed by Eq. (A.12), when $k=3$ and $q=1$ (i.e. $t\to\infty$), we have the following inequality

$$\begin{aligned}
& D(\{0,0,c_3\},\{0,0,0\}) \\
= \;& D(\Lambda_1^{F_3\otimes N}\{c_1,(-1)^{N/2}c_1c_3,c_3\},\Lambda_1^{F_3\otimes N}\{c_1,0,0\}) \qquad (\mathrm{A.21})\\
\le \;& D(\{c_1,(-1)^{N/2}c_1c_3,c_3\},\{c_1,0,0\}),
\end{aligned}$$

where the first equality is due to the fact that

$$\begin{aligned}
\{0,0,c_3\} &= \Lambda_1^{F_3\otimes N}\{c_1,(-1)^{N/2}c_1c_3,c_3\}, \text{ and} \qquad (\mathrm{A.22})\\
\{0,0,0\} &= \Lambda_1^{F_3\otimes N}\{c_1,0,0\}, \qquad (\mathrm{A.23})
\end{aligned}$$

while the final inequality in (A.21) is due to the contractivity of the distance D.

In order to prove the opposite inequality and thus Eq. (A.19), we now introduce a N-qubit global *rephasing* channel $\Lambda_r^{R_3}$ which is defined in the operator-sum representation as

$$\Lambda_r^{R_3}(\rho) = \sum_{i,\pm} K_{i,\pm}^{R_3}\rho K_{i,\pm}^{R_3\,\dagger}, \tag{A.24}$$

with

$$\begin{aligned}
K_{1,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_1^\pm\rangle\langle 000\dots000|, \qquad (\mathrm{A.25})\\
K_{2,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_2^\pm\rangle\langle 000\dots001|,\\
K_{3,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_3^\pm\rangle\langle 000\dots010|,\\
K_{4,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_4^\pm\rangle\langle 000\dots011|,\\
&\dots\\
K_{2^{N-1}-1,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_{2^{N-1}-1}^\pm\rangle\langle 011\dots110|,\\
K_{2^{N-1},\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_{2^{N-1}}^\pm\rangle\langle 011\dots111|,\\
K_{2^{N-1}+1,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_{2^{N-1}}^\pm\rangle\langle 100\dots000|,\\
K_{2^{N-1}+2,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_{2^{N-1}-1}^\pm\rangle\langle 100\dots001|,\\
&\dots\\
K_{2^N-3,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_4^\pm\rangle\langle 111\dots100|,\\
K_{2^N-2,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_3^\pm\rangle\langle 111\dots101|,\\
K_{2^N-1,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_2^\pm\rangle\langle 111\dots110|,\\
K_{2^N,\pm}^{R_3} &= \sqrt{\frac{1\pm r}{2}}|\beta_1^\pm\rangle\langle 111\dots111|,
\end{aligned}$$

where $r\in[0,1]$ is a parameter denoting the rephasing strength, $\{|\beta_i^\pm\rangle\}$ is the N-qubit basis defined in Eq. (A.14), and the 2^{N+1} Kraus operators satisfy $\sum_{i,\pm}K_{i,\pm}^{R_3\,\dagger}K_{i,\pm}^{R_3}=\mathbb{I}^{\otimes N}$, thus ensuring that $\Lambda_r^{R_3}$ is a CPTP map.

It is now essential to see that the effect of $\Lambda_r^{R_3}$ on a M_N^3 state of the form $\{0,0,c_3\}$ is given by

$$\Lambda_r^{R_3}(\{0,0,c_3\}) = \{r,(-1)^{N/2}r\,c_3,c_3\}, \tag{A.26}$$

for any even N. To prove Eq. (A.26), it will be useful to split the N-qubit states $|\beta_i^\pm\rangle$ into the states $|\Phi_i^\pm\rangle$ and $|\Psi_i^\pm\rangle$ with even and odd parity, respectively, i.e. such that

$$\begin{aligned}
\Pi_3|\Phi_i^\pm\rangle &= |\Phi_i^\pm\rangle,\\
\Pi_3|\Psi_i^\pm\rangle &= -|\Psi_i^\pm\rangle, \qquad (\mathrm{A.27})
\end{aligned}$$

where $i\in\{1,\cdots,2^{N-2}\}$. Thanks to Eqs. (A.15), (A.16) (A.17) and (A.27), one gets that the spectral decomposition of a M_N^3 state $\rho_{\{c_1,c_2,c_3\}}$ with generic triple $\{c_1,c_2,c_3\}$ can be written as follows,

$$\begin{aligned}
&\rho_{\{c_1,c_2,c_3\}} \qquad (\mathrm{A.28})\\
&= \frac{1}{2^N}\left[1+c_1+(-1)^{N/2}c_2+c_3\right]\sum_i|\Phi_i^+\rangle\langle\Phi_i^+|\\
&+ \frac{1}{2^N}\left[1-c_1-(-1)^{N/2}c_2+c_3\right]\sum_i|\Phi_i^-\rangle\langle\Phi_i^-|\\
&+ \frac{1}{2^N}\left[1+c_1-(-1)^{N/2}c_2-c_3\right]\sum_i|\Psi_i^+\rangle\langle\Psi_i^+|\\
&+ \frac{1}{2^N}\left[1-c_1+(-1)^{N/2}c_2-c_3\right]\sum_i|\Psi_i^-\rangle\langle\Psi_i^-|
\end{aligned}$$

As a consequence

$$\begin{aligned}
&\rho_{\{0,0,c_3\}} \qquad (\mathrm{A.29})\\
&= \frac{1}{2^N}(1+c_3)\sum_i|\Phi_i^+\rangle\langle\Phi_i^+|\\
&+ \frac{1}{2^N}(1+c_3)\sum_i|\Phi_i^-\rangle\langle\Phi_i^-|\\
&+ \frac{1}{2^N}(1-c_3)\sum_i|\Psi_i^+\rangle\langle\Psi_i^+|\\
&+ \frac{1}{2^N}(1-c_3)\sum_i|\Psi_i^-\rangle\langle\Psi_i^-|,
\end{aligned}$$

while

$$\begin{aligned} &\rho_{\{r,(-1)^{N/2}r\ c_3,c_3\}} && \text{(A.30)}\\ &= \frac{1}{2^N}(1+r)(1+c_3)\sum_i |\Phi_i^+\rangle\langle\Phi_i^+| \\ &+ \frac{1}{2^N}(1-r)(1+c_3)\sum_i |\Phi_i^-\rangle\langle\Phi_i^-| \\ &+ \frac{1}{2^N}(1+r)(1-c_3)\sum_i |\Psi_i^+\rangle\langle\Psi_i^+| \\ &+ \frac{1}{2^N}(1-r)(1-c_3)\sum_i |\Psi_i^-\rangle\langle\Psi_i^-|. \end{aligned}$$

By exploiting the following equalities

$$\Lambda_r^{R_3}(|\Phi_i^+\rangle\langle\Phi_i^+|) = \frac{1+r}{2}|\Phi_i^+\rangle\langle\Phi_i^+| + \frac{1-r}{2}|\Phi_i^-\rangle\langle\Phi_i^-|, \quad \text{(A.31)}$$
$$\Lambda_r^{R_3}(|\Phi_i^-\rangle\langle\Phi_i^-|) = \frac{1+r}{2}|\Phi_i^+\rangle\langle\Phi_i^+| + \frac{1-r}{2}|\Phi_i^-\rangle\langle\Phi_i^-|,$$
$$\Lambda_r^{R_3}(|\Psi_i^+\rangle\langle\Psi_i^+|) = \frac{1+r}{2}|\Psi_i^+\rangle\langle\Psi_i^+| + \frac{1-r}{2}|\Psi_i^-\rangle\langle\Psi_i^-|,$$
$$\Lambda_r^{R_3}(|\Psi_i^-\rangle\langle\Psi_i^-|) = \frac{1+r}{2}|\Psi_i^+\rangle\langle\Psi_i^+| + \frac{1-r}{2}|\Psi_i^-\rangle\langle\Psi_i^-|,$$

and the linearity of the global rephasing channel, we get

$$\begin{aligned} \Lambda_r^{R_3}(\{0,0,c_3\}) &= \frac{1}{2^N}(1+c_3)\sum_i \Lambda_r^{R_3}(|\Phi_i^+\rangle\langle\Phi_i^+|) \quad \text{(A.32)}\\ &+ \frac{1}{2^N}(1+c_3)\sum_i \Lambda_r^{R_3}(|\Phi_i^-\rangle\langle\Phi_i^-|) \\ &+ \frac{1}{2^N}(1-c_3)\sum_i \Lambda_r^{R_3}(|\Psi_i^+\rangle\langle\Psi_i^+|) \\ &+ \frac{1}{2^N}(1-c_3)\sum_i \Lambda_r^{R_3}(|\Psi_i^-\rangle\langle\Psi_i^-|) \\ &= \{r,(-1)^{N/2}r\ c_3,c_3\}. \end{aligned}$$

We then have the inequality

$$\begin{aligned} &D(\{c_1,(-1)^{N/2}c_1c_3,c_3\},\{c_1,0,0\}) \\ &= D(\Lambda_{c_1}^{R_3}\{0,0,c_3\},\Lambda_{c_1}^{R_3}\{0,0,0\}) \quad \text{(A.33)}\\ &\le D(\{0,0,c_3\},\{0,0,0\}), \end{aligned}$$

where the first equality is due to the fact that

$$\begin{aligned} \{c_1,(-1)^{N/2}c_1c_3,c_3\} &= \Lambda_{c_1}^{R_3}\{0,0,c_3\}, \text{ and} \\ \{c_1,0,0\} &= \Lambda_{c_1}^{R_3}\{0,0,0\}, \end{aligned}$$

while the final inequality in (A.33) is again due to the contractivity of the distance D. By putting together the two inequalities (A.21) and (A.33), we immediately get the invariance of Eq. (A.19) for any contractive distance.

In order now to prove Eq. (A.20), we introduce the local unitary $V^{\otimes N}$ with $V = \frac{1}{\sqrt{2}}(\mathbb{I} + i\sigma_2)$. The effect of $V^{\otimes N}$ on a general M_N^3 state is given by

$$V^{\otimes N}\{c_1,c_2,c_3\}V^{\otimes N\dagger} = \{c_3,c_2,c_1\}, \quad \text{(A.34)}$$

where this can be easily seen by utilising the fact that N is even and the following single-qubit identities:

$$\begin{aligned} V\sigma_1V^\dagger &= \sigma_3, \\ V\sigma_2V^\dagger &= \sigma_2, \\ V\sigma_3V^\dagger &= -\sigma_1. \end{aligned}$$

Thanks to the invariance under unitaries of any contractive distance D, the effect of the unitary $V^{\otimes N}$ expressed by Eq. (A.34), and the just proven invariance expressed by Eq. (A.19), we eventually have

$$\begin{aligned} &D(\{c_1,(-1)^{N/2}c_1c_3,c_3\},\{0,0,c_3\}) \quad \text{(A.35)}\\ &= D(V^{\otimes N}\{c_1,(-1)^{N/2}c_1c_3,c_3\}V^{\otimes N\dagger}, V^{\otimes N}\{0,0,c_3\}V^{\otimes N\dagger}) \\ &= D(\{c_3,(-1)^{N/2}c_1c_3,c_1\},\{c_3,0,0\}) \\ &= D(\{0,0,c_1\},\{0,0,0\}) \\ &= D(V^{\otimes N}\{0,0,c_1\}V^{\otimes N\dagger}, V^{\otimes N}\{0,0,0\}V^{\otimes N\dagger}) \\ &= D(\{c_1,0,0\},\{0,0,0\}), \end{aligned}$$

that is Eq. (A.20).

■

Lemma A.2. *For all even N, according to any contractive and convex distance D, one of the closest incoherent states δ_ρ to a M_N^3 state ρ is always a M_N^3 incoherent state, i.e. one of the form*

$$\delta_\rho = \frac{1}{2^N}\left(\mathbb{I}^{\otimes N} + s\,\sigma_3^{\otimes N}\right) \quad \text{(A.36)}$$

for some coefficient $s \in [-1,1]$.

Proof. Consider an arbitrary N-qubit state ρ, which can be represented as

$$\rho = \frac{1}{2^N}\sum_{i_1,i_2,\ldots,i_N=0}^{3} \tau_{i_1i_2\ldots i_N}\sigma_{i_1}\otimes\sigma_{i_2}\ldots\otimes\sigma_{i_N}, \quad \text{(A.37)}$$

where the coefficients $\tau_{i_1,i_2,\ldots i_N} = \mathrm{Tr}\left[\rho\,\sigma_{i_1}\otimes\sigma_{i_2}\ldots\otimes\sigma_{i_N}\right] \in [-1,1]$ are the correlation tensor elements of ρ, and $\sigma_0 \equiv \mathbb{I}$. Any term involving σ_1 or σ_2 in the tensorial sum (A.37) introduces off-diagonal elements, therefore we can write a general N-qubit incoherent state, with respect to the computational basis, as

$$\delta = \frac{1}{2^N}\sum_{i_1,i_2,\ldots,i_N=\{0,3\}} \tau_{i_1i_2\ldots i_N}\sigma_{i_1}\otimes\sigma_{i_2}\ldots\otimes\sigma_{i_N}, \quad \text{(A.38)}$$

where each index i_j can now take either 0 or 3 as the only values. For any N-qubit incoherent state δ, we can define a corresponding incoherent M_N^3 state $\delta_{\mathrm{M}_N^3}$, whose τ tensor is obtained from the one of δ by setting all the $\tau_{i_1,i_2,\ldots i_N}$ equal to zero, but for the two entries $\tau_{00\ldots0}$ and $\tau_{33\ldots3}$. We want to show that $D(\rho, \delta_{\mathrm{M}_N^3}) \leq D(\rho, \delta)$ for any M_N^3 state ρ and N-qubit incoherent state δ, which readily implies that one of the closest incoherent states δ_ρ to a M_N^3 state ρ is indeed a M_N^3 state.

To begin, first consider the family of $N-1$ unitaries

$$\begin{aligned}\{U_j\}_{j=1}^{N-1} = \{&(\sigma_1 \otimes \sigma_1 \otimes \mathbb{I}^{\otimes N-2}), (\mathbb{I} \otimes \sigma_1 \otimes \sigma_1 \otimes \mathbb{I}^{\otimes N-3}), \\ &(\mathbb{I}^{\otimes 2} \otimes \sigma_1 \otimes \sigma_1 \otimes \mathbb{I}^{\otimes N-4}), (\mathbb{I}^{\otimes 3} \otimes \sigma_1 \otimes \sigma_1 \otimes \mathbb{I}^{\otimes N-5}), \\ &\ldots, (\mathbb{I}^{\otimes N-3} \otimes \sigma_1 \otimes \sigma_1 \otimes \mathbb{I}), (\mathbb{I}^{\otimes N-2} \otimes \sigma_1 \otimes \sigma_1)\}.\end{aligned} \tag{A.39}$$

We note that every M_N^3 state ρ is invariant under the action of any U_j. This can be seen as follows

$$\begin{aligned} U_j \rho U_j^\dagger &= \frac{1}{2^N}\left(U_j \mathbb{I}^{\otimes N} U_j^\dagger + \sum_{i=1}^{3} c_i U_j \sigma_i^{\otimes N} U_j^\dagger\right) \\ &= \frac{1}{2^N}\left(\mathbb{I}^{\otimes N} + \sum_{i=1}^{3} c_i \sigma_i^{\otimes N}\right) = \rho, \end{aligned} \tag{A.40}$$

where in the second equality we use $U_j \mathbb{I}^{\otimes N} U_j^\dagger = \mathbb{I}^{\otimes N}$ and $U_j \sigma_i^{\otimes N} U_j^\dagger = \sigma_i^{\otimes N}$ which arises simply by recalling $\sigma_1\sigma_1\sigma_1 = \sigma_1$, $\sigma_1\sigma_2\sigma_1 = -\sigma_2$ and $\sigma_1\sigma_3\sigma_1 = -\sigma_3$ and noting that there are always two σ_1's in each unitary.

Now consider the action of U_1 on a generic incoherent state δ. The state transforms as

$$U_1 \delta U_1^\dagger = \frac{1}{2^N} \sum_{i_1, i_2, \ldots, i_N = \{0,3\}} \tau_{i_1 i_2 \ldots i_N} \sigma_1 \sigma_{i_1} \sigma_1 \otimes \sigma_1 \sigma_{i_2} \sigma_1 \otimes \sigma_{i_3} \otimes \ldots \otimes \sigma_{i_N}. \tag{A.41}$$

We have $\sigma_1\sigma_0\sigma_1 = \sigma_0$ and $\sigma_1\sigma_3\sigma_1 = -\sigma_3$, hence the coefficients $\tau_{i_1 i_2 \ldots i_N}^{U_1}$ of $U_1 \delta U_1^\dagger$ are $\tau_{00\ldots}^{U_1} = \tau_{00\ldots}$, $\tau_{33\ldots}^{U_1} = \tau_{33\ldots}$, $\tau_{03\ldots}^{U_1} = -\tau_{03\ldots}$ and $\tau_{30\ldots}^{U_1} = -\tau_{30\ldots}$; in other words, U_1 flips the sign of any element $\tau_{i_1 i_2 \ldots i_N}$ for which $i_1 \neq i_2$. We can further define a state that is a linear combination of δ and $U_1 \delta U_1^\dagger$,

$$\delta^1 = \frac{1}{2}(\delta + U_1 \delta U_1^\dagger). \tag{A.42}$$

The coefficients $\tau_{i_1 i_2 \ldots i_N}^1$ of δ^1 can be found simply as

$$\tau_{i_1 i_2 \ldots i_N}^1 = \frac{1}{2}\left(\tau_{i_1 i_2 \ldots i_N} + \tau_{i_1 i_2 \ldots i_N}^{U_1}\right). \tag{A.43}$$

We see therefore that $\tau_{00\ldots}^1 = \tau_{00\ldots}$, $\tau_{33\ldots}^1 = \tau_{33\ldots}$, $\tau_{03\ldots}^1 = 0$ and $\tau_{30\ldots}^1 = 0$.

Now, convexity and contractivity of the distance D can be used to establish the inequality $D(\rho, \delta^1) \leq D(\rho, \delta)$ for any M_N^3 state ρ and any incoherent state δ. Indeed,

$$\begin{aligned} D(\rho, \delta^1) &= D\left(\rho, \frac{1}{2}(\delta + U_1 \delta U_1^\dagger)\right) \\ &\leq \frac{1}{2}\left(D(\rho, \delta) + D(\rho, U_1 \delta U_1^\dagger)\right) \\ &= \frac{1}{2}\left(D(\rho, \delta) + D(U_1 \rho U_1^\dagger, U_1 \delta U_1^\dagger)\right) \\ &= D(\rho, \delta), \end{aligned} \tag{A.44}$$

where in the first equality we use the definition of δ^1, in the subsequent inequality we use the convexity of D, in the equality on the third line we use the invariance of ρ through U_1, i.e. $U_1 \rho U_1^\dagger = \rho$, and in the final equality we use the invariance of D through unitaries $D(U_1 \rho U_1^\dagger, U_1 \delta U_1^\dagger) = D(\rho, \delta)$ implied by the contractivity of D.

Returning to the action of U_j on δ, it is a simple extension of the previous argument for U_1 to see that U_j flips the value of $\tau_{i_1 i_2 \ldots i_N}$ when $i_j \neq i_{j+1}$. We are now in a position to define a set of incoherent states $\{\delta^0, \delta^1, \delta^2 \ldots \delta^{N-1}\}$ in an iterative way

$$\delta^j = \frac{1}{2}\left(\delta^{j-1} + U_j \delta^{j-1} U_j^\dagger\right), \tag{A.45}$$

for $j \in [1, N-1]$ and $\delta^0 \equiv \delta$. The initial state is the incoherent state δ with correlation tensor elements $\tau_{i_1 i_2 \ldots i_N}$. The first state δ^1 loses all the $\tau_{i_1 i_2 \ldots i_N}$ from δ for which $i_1 \neq i_2$. Next, the j-th state δ^j loses all the $\tau_{i_1 i_2 \ldots i_N}$ from δ for which $i_j \neq i_{j+1}$. The final state δ^{N-1} loses all the $\tau_{i_1 i_2 \ldots i_N}$ from δ for which $i_{N-1} \neq i_N$. The only remaining values of $\tau_{i_1 i_2 \ldots i_N}$ from δ in δ^{N-1} are those for which $i_1 = i_2 = i_3 \ldots = i_N$. Only $\tau_{00\ldots0}$ and $\tau_{33\ldots3}$ obey this condition, i.e. δ^{N-1} is the incoherent M_N^3 state $\delta_{\mathrm{M}_N^3}$.

The inequality $D(\rho, \delta^1) \leq D(\rho, \delta)$ can now be generalised iteratively,

$$\begin{aligned} D(\rho, \delta^j) &= D\left(\rho, \frac{1}{2}(\delta^{j-1} + U_j \delta^{j-1} U_j^\dagger)\right) \\ &\leq \frac{1}{2}\left(D(\rho, \delta^{j-1}) + D(\rho, U_j \delta^{j-1} U_j^\dagger)\right) \\ &= \frac{1}{2}\left(D(\rho, \delta^{j-1}) + D(U_j \rho U_j^\dagger, U_j \delta^{j-1} U_j^\dagger)\right) \\ &= D(\rho, \delta^{j-1}), \end{aligned} \tag{A.46}$$

where we use, in order, the definition of δ^j for $j \in [1, N-1]$, the convexity of D, the invariance of ρ through any U_j, i.e. $U_j \rho U_j^\dagger = \rho$, and the invariance of D through unitaries, $D(U_j \rho U_j^\dagger, U_j \delta^{j-1} U_j^\dagger) = D(\rho, \delta^{j-1})$.

This process gives a hierarchy of $N-1$ inequalities $D(\rho, \delta^j) \leq D(\rho, \delta^{j-1})$, which chained together imply $D(\rho, \delta^0) \leq D(\rho, \delta^{N-1})$. We know that $\delta^0 \equiv \delta$ and $\delta^{N-1} \equiv \delta_{\mathrm{M}_N^3}$, hence we have shown that

$$D(\rho, \delta_{\mathrm{M}_N^3}) \leq D(\rho, \delta) \quad \forall \delta. \tag{A.47}$$

■

Lemma A.3. *For all even N, according to any contractive distance D, it holds that one of the closest incoherent* M_N^3 *states* δ *with triple* $\{0, 0, s\}$ *to a* M_N^3 *state* ρ *with triple* $\{c_1, (-1)^{N/2} c_1 c_3, c_3\}$ *is specified by* $s = c_3$.

Proof. We need to prove that, for any z, it holds that

$$\begin{aligned} &D(\{c_1, (-1)^{N/2} c_1 c_3, c_3\}, \{0, 0, c_3\}) \\ &\leq D(\{c_1, (-1)^{N/2} c_1 c_3, c_3\}, \{0, 0, c_3 + z\}). \end{aligned} \tag{A.48}$$

In fact ■

$$\begin{aligned} & D(\{c_1, (-1)^{N/2} c_1 c_3, c_3\}, \{0, 0, c_3\}) \\ = \ & D(\{c_1, 0, 0\}, \{0, 0, 0\}) \\ = \ & D(\Lambda_1^{F_1 \otimes N}\{c_1, (-1)^{N/2} c_1 c_3, c_3\}, \Lambda_1^{F_1 \otimes N}\{0, 0, c_3 + z\}) \\ \leq \ & D(\{c_1, (-1)^{N/2} c_1 c_3, c_3\}, \{0, 0, c_3 + z\}), \end{aligned}$$

where the first equality is due to Lemma A.1, which holds for any contractive distance D and any even N, the second equality is due to the fact that

$$\{c_1, 0, 0\} = \Lambda_1^{F_1 \otimes N}\{c_1, (-1)^{N/2} c_1 c_3, c_3\}, \text{ and} \quad (A.49)$$

$$\{0, 0, 0\} = \Lambda_1^{F_1 \otimes N}\{0, 0, c_3 + z\}, \quad (A.50)$$

with $\Lambda_1^{F_1 \otimes N}$ representing the action of N local independent bit flip noisy channels expressed by Eq. (A.12), when $k = 1$ and $q = 1$ (i.e., $t \to \infty$), and finally the inequality is due to contractivity of the distance D.

Due to Lemma A.2 and Lemma A.3, we finally get that any bona fide distance-based measure of quantum coherence C_D of the evolved M_N^3 state $\rho(q)$, given in Eq. (A.13), is equal to the following distance

$$\begin{aligned} C_D(\rho(q)) = \ & D(\{c_1, (-1)^{N/2}(1-q)^N c_1 c_3, (1-q)^N c_3\}, \\ & \{0, 0, (1-q)^N c_3\}), \end{aligned} \quad (A.51)$$

which is frozen for any q (equivalently, for any time t) thanks to Lemma A.1, Eq. (A.20). This concludes the proof of the central result in the main text.

Notice further that Lemma A.3 implies that the l_1-norm of coherence equals (twice) the trace distance of coherence for all M_N^3 states with even N, which entails that C_{l_1} is frozen as well in the same dynamical conditions as for all bona fide distance-based measures of coherence, including e.g. C_{RE}.