A recursive approach for geometric quantifiers of pairwise quantum discord in multiqubit Shrödinger cat states

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Abstract

A recursive approach for determining the Hilbert-Schmidt measure of pairwise quantum discord in a balanced superposition of symmetric n-qubit states (multiqubit Shrödinger cat states) is presented. A particular emphasis is devoted to k qubits reduced states obtained by tracing out $n-k$ particles ($k = 2, 3, \cdots, n-1$) from the whole system. Two bi-partition schemes are discussed. One consists in dividing the k-qubits in two subsystems containing one and $(k-1)$ qubits respectively. The explicit derivation of the pairwise geometric discord uses recursive relations between the Fano-Bloch correlation matrices. A detailed analysis is given for two, three and four qubit systems. In the second scheme, the cluster comprising the $(k-1)$ qubits is mapped into a system of two logical qubits. We show that these two bi-partitioning schemes are equivalent in evaluating the pairwise geometric discord in multiqubit Shrödinger cat states. The explicit expressions of classical states exhibiting zero discord are also derived.

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1 Introduction

Quantum correlations in multipartite systems have generated a lot of interest during the last two decades [1, 2, 3]. This is essentially motivated by their promising applications in the field of quantum information that guarantee classically unattainable efficiency in a variety of quantum tasks (see for instance [4, 5]). In this sense, the characterization of quantum correlations is crucial to implement quantum protocols such as quantum teleportation [6], superdense coding [7] and quantum key distribution [8]. Several methods and different measures of quantum correlation were exhaustively discussed in the literature from various perspectives and for many purposes (for a recent review see [3]). They can be classified in two main categories: entropic based measures and geometric quantifiers or norm based measures. Entanglement of formation, linear entropy, relative entropy and quantum discord [9, 10, 11, 12, 13, 14] constitute familiar entropic quantifiers of correlations. Probably, quantum discord, which goes beyond entanglement in distinguishing between quantum and classical states, is the most prominent of these correlations. It was originally defined as the difference between two quantum analogues of the classical mutual information [13, 14]. The explicit evaluation of based entropy measures requires optimization procedures which are in general very complicated to achieve and constitute the main obstacle in order to get computable expressions of quantum correlations. To overcome such difficulties, geometric measures, especially ones based on Hilbert-Schmidt norm, were considered to formulate a geometric variant of quantum discord [15]. The Hilbert-Schmidt distance was used also to quantify classical correlations [16, 17]. More recently, it has been pointed that 1-norm distance (trace distance) offers also a geometric tool to derive the amount of quantum and classical correlations in bipartite systems [18, 19, 20, 21].

In other hand, the extension of Hilbert-Schmidt measure of quantum discord to d-dimensional quantum systems (qudits) was reported in [22, 23, 24] (see also [25] and references quoted therein). Interestingly, this higher dimensional extension can be adapted to evaluate the pairwise quantum correlations in multi-qubit systems. In fact in multi-qubit systems, geometric quantum discord based on the Hilbert-Schmidt norm is more tractable from a computational point of view than entropic based measures. In this spirit, we shall employ in this work the approach proposed by Dakic et al [15] to investigate the quantum correlations in mixed multi-qubit states. Specifically, we will consider a balanced superposition of symmetric multi-qubit states in which the symmetry properties offer drastic simplification in evaluating quantum correlations.

This paper is organized as follows. In section 2, we discuss the relevance of symmetric multi-qubit (n-qubits) states in defining Schrödinger cat states. We shall essentially focus on balanced superpositions, symmetric or antisymmetric under parity transformation, which coincide with even and odd spin atomic coherent states. A special attention, in section 3, is devoted to reduced states describing subsystems containing k qubits ($k = 2, 3, \cdots, n-1$) obtained by tracing out $(n-k)$ qubits from a

n-qubit Schrödinger cat state. The obtained reduced density matrices are X shaped and subsequently their algebraic structures simplify considerably the determination the quantum correlation based on Hilbert-Schmidt (geometric quantum discord) between one qubit and $(k-1)$ qubits contained in a mixed k-qubit state. This is explicitly described in section 4. In particular considering the cases of two and three qubit systems, we provide the general method to determine analytically geometric discord in mixed k-qubits states. We also derive the explicit forms of classical (zero discord) states. In section 5, we introduce another scheme according to which the second part of the system comprising $(k-1)$ qubits is regarded as a single qubit. In this picture the whole system reduces to a two qubit system. Remarkably, the geometric measure of quantum discord obtained, in this second scheme, coincides with one derived in the first bi-partition scheme (section 4). As illustration, a detailed analysis is given for $k=3$ and $k=4$. The method developed in this paper which extends the geometric measure of two-qubit X states to embrace k-qubit X states is useful in investigating the global pairwise correlation in multipartite qubit systems. Concluding remarks close this paper.

2 Symmetric multi-qubit systems

The multi-qubit symmetric states were shown relevant for different purposes in quantum information science [26, 27, 28, 29, 30, 31, 32, 33]. In this paper, we shall mainly focus on an ensemble of n spin-1/2 prepared in even and odd spin coherent states.

2.1 Spin coherent as symmetric multi-qubit systems

We consider n identical qubits. Each qubit lives in a 2-dimensional Hilbert space $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$. The Hilbert space of the n-qubit system is given by n tensored copies of $\mathcal{H}$

$$\mathcal{H}_n := \mathcal{H}^{\otimes n}.$$

Among the multi-partite states in $\mathcal{H}_n$, multi-qubit states obeying exchange symmetry are of special interest from experimental as well as mathematical point of views. An arbitrary symmetric n-qubit state is commonly represented in either Majorana [34] or Dicke [35] representation. Any multi-qubit state, invariant under the exchange symmetry, is specified in the Majorana description by the state (up to a normalization factor)

$$|\psi_s\rangle = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} |\eta_{\sigma(1)}, \dots, \eta_{\sigma(n)}\rangle, \tag{1}$$

where each single qubit state is $|\eta_i\rangle \equiv (1 + \eta_i \bar{\eta}_i)^{-\frac{1}{2}}(|0\rangle + \eta_i |1\rangle)$ $(i = 1, \dots, n)$ and the sum is over the elements of the permutation group $\mathcal{S}_n$ of n objects. In Equation (1), the vector $|\eta_{\sigma(1)}, \dots, \eta_{\sigma(n)}\rangle$ stands for the tensor product $|\eta_{\sigma(1)}\rangle \otimes \dots \otimes |\eta_{\sigma(n)}\rangle$. The totally symmetric n-qubit states can be also formulated in Dike representation. The symmetric Dicke states with k excitations are defined by [35]

$$|n, k\rangle = \sqrt{\frac{k!(n-k)!}{n!}} \sum_{\sigma \in \mathcal{S}_n} |\underbrace{0, \dots, 0}_{n-k}, \underbrace{1, \dots, 1}_{k}\rangle, \tag{2}$$

which generate an orthonormal basis of the symmetric Hilbert subspace of dimension $(n+1)$. Therefore, permutation invariance, in symmetric multi-qubit states, implies a restriction to $n+1$ dimensional subspace from the entire 2^n dimensional Hilbert space. The Dicke states (2) constitute a special subset of the symmetric multi-qubit states (1) corresponding to the situation where the first k-qubit are such that $\eta_i = 0$ for $i = 0, 1, \ldots k$ and the remaining qubits are in the states $|\eta_i = 1\rangle$ with $i = k+1, \ldots, n$. Any symmetric state $|\psi_s\rangle$ (1) can be expanded in terms of Dicke states (2) as follows

$$|\psi_s\rangle = \frac{1}{n!}\sum_{k=0}^{n} c_k \; |n,k\rangle, \tag{3}$$

where the c_k's ($k = 0, \ldots, n$) stand for complex expansion coefficients. In particular, when the qubits are all identical ($\eta_i = \eta$ for all qubits), it is simple to check that the coefficients c_k are given by

$$c_k = n!\sqrt{\frac{n!}{k!(n-k)!}}\frac{\eta^k}{(1+\eta\bar{\eta})^{\frac{n}{2}}} \tag{4}$$

and the symmetric multi-qubit states (1) write

$$|\psi_s\rangle := |n,\eta\rangle = (1+\eta\bar{\eta})^{-\frac{n}{2}}\sum_{k=0}^{n}\sqrt{\frac{n!}{k!(n-k)!}}\eta^k \; |n,k\rangle, \tag{5}$$

which are exactly the $j = \frac{n}{2}$-spin coherent states (for more details see for instance [36]). In particular, the state $|n,\eta\rangle$ can be identified for $n = 1$ with spin-$\frac{1}{2}$ coherent state with $|0\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|1\rangle \equiv |\frac{1}{2}, +\frac{1}{2}\rangle$)

2.2 Multi-qubit "Shroödinger cat" states

The prototypical multi-qubit "Shrödinger cat" states, the basic objects in this work, are defined as a balanced superpositions of the n-qubit states $|n,\eta\rangle$ and $|n,-\eta\rangle$ given by (5). They write

$$|\eta, n, m\rangle = \mathcal{N}(|n,\eta\rangle + e^{im\pi}|n,-\eta\rangle) \tag{6}$$

where

$$|n, \pm\eta\rangle = |\pm\eta\rangle \otimes |\pm\eta\rangle \cdots \otimes |\pm\eta\rangle,$$

and the integer $m \in \mathbb{Z}$ takes the values $m = 0$ (mod 2) and $m = 1$ (mod 2). The normalization factor $\mathcal{N}$ is

$$\mathcal{N} = \left[2 + 2p^n \cos m\pi\right]^{-1/2}$$

where p denotes the overlap between the states $|\eta\rangle$ and $|-\eta\rangle$. It is given by

$$p = \langle\eta| - \eta\rangle = \frac{1-\bar{\eta}\eta}{1+\bar{\eta}\eta}. \tag{7}$$

Experimental creation of cat states comprising multiple particles was reported in the literature [37, 38]. Due to their experimental implementation, "Shrödinger cat" states are expected to serve as an useful resource for quantum computing as well as quantum communications. Also, in view of their mathematical elegance, multi-qubit states obeying exchange symmetry offer simplification in investigating various aspects of quantum correlations in particular the geometric measure of quantum discord as we shall discuss in the present work. Furthermore, the multi-qubit symmetric states (6) include Greenberger-Horne-Zeilinger(GHZ) [39], W [40] and Dicke states [35]. The multi-qubits states $|n, \eta, 0\rangle$ ($m = 0$ mod 2) and $|n, \eta, 1\rangle$ ($m = 1$ mod 2) behave like a multipartite state of Greenberger-Horne-Zeilinger (GHZ) type [39] in the limiting case $p \to 0$. Indeed, the states $|\eta\rangle$ and $|-\eta\rangle$ approach orthogonality and an orthogonal basis can be defined such that $|\mathbf{0}\rangle \equiv |\eta\rangle$ and $|\mathbf{1}\rangle \equiv |-\eta\rangle$. Thus, the state $|n, \eta, m\rangle$ becomes of GHZ-type:

$$|\eta, n, m\rangle \sim |\mathrm{GHZ}\rangle_n = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \cdots \otimes |\mathbf{0}\rangle + e^{im\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes \cdots \otimes |\mathbf{1}\rangle). \tag{8}$$

Also, in the special situation where the overlap p tends to unity ($p \to 1$ or $\eta \to 0$), the state $|\eta, n, m = 0 \ (\mathrm{mod}\ 2)\rangle$ (6) reduces to ground state of a collection of n qubits

$$|0, n, 0 \ (\mathrm{mod}\ 2)\rangle \sim |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle, \tag{9}$$

and it is simple to check that the state $|\eta, 0, 1 \ (\mathrm{mod}\ 2)\rangle$ becomes a multipartite state of W type [40]

$$|0, n, 1 \ (\mathrm{mod}\ 2)\rangle \sim |\mathrm{W}\rangle_n = \frac{1}{\sqrt{n}}(|1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes \ldots \otimes |0\rangle + \cdots + |0\rangle \otimes |0\rangle \otimes \cdots \otimes |1\rangle) \ . \tag{10}$$

Hence, the Shrödinger cat states $|\eta, n, m = 0 \ (\mathrm{mod}\ 2)\rangle$ include the $|\mathrm{GHZ}\rangle_n$ states ($p \to 0$). In other hand, the states $|\eta, n, m = 1 \ (\mathrm{mod}\ 2)\rangle$, constitute an interpolation between two special classes of multi-qubits states: $|\mathrm{GHZ}\rangle_n$ type corresponding to $p \to 0$ and states of $|\mathrm{W}\rangle_n$ type obtained in the special case where $p \to 1$.

3 Quantum correlations in multi-partite Shrödinger cat states.

The structure of multipartite correlations within multi-qubit quantum systems is a challenging and daunting task. With the growth of number of qubits, there are numerous ways in splitting the entire system to characterize how the particles are correlated. Obviously, the bipartite splitting of the whole system is not sufficient to capture the essential of quantum correlation existing in a multi-qubit system. However, it must be stressed that the pairwise decomposition of total correlation is unavoidable and offers a good alternative to evaluate the amount of all correlations existing in a multipartite system. In this paper, we approach the problem of analyzing n-qubit correlation using only bipartite measures. Toward this end, we consider first the correlation between one qubit with the remaining $(n-1)$ qubits in the state (6). Thus, the pure density matrix of the symmetric n-qubit system writes

$$\rho_n \equiv |\eta, n, m\rangle\langle\eta, n, m| := \rho_{1|23\ldots n}.$$

Furthermore, after removing $k = 1, 2, \cdots, n-2$ particles from the n-qubit system, the reduced density matrix ρ_{n-k} can be bi-partitioned in two subsystem, one comprises one qubit and the remaining $(n-k-1)$ qubits are contained in the second subsystem. In this way, after identifying the two subsystems, one can proceed with the analytical evaluation of the pairwise quantum correlation existing among them. Also, this scheme offers a reasonable method to characterize the total amount of quantum correlation defined as the sum of the quantum correlations for all possible bi-partitions [41]. In this paper, we shall employ this picture to estimate the geometric measure of quantum discord (D_g) in the symmetric multi-qubit system of the form (6). We give a detailed analysis for two qubit and three qubit subsystems. From these two specific cases, we give a general algorithm to determine recursively the pairwise quantum discord in a reduced density describing k qubit system.

3.1 Two-qubit states

The strategy and tools we introduce for a two-qubit subsystem are useful when extending the size of the system to encompass more qubits. We consider the two-qubit states extracted from the state (6) by tracing out $(n-2)$ qubits. Since the n qubits are all identical, we obtain $n(n-1)/2$ identical density matrices. They are given by

$$\rho_{12} = \mathcal{N}^2 \bigg[|\eta, \eta\rangle\langle\eta, \eta| + e^{im\pi} q_2 |-\eta, -\eta\rangle\langle\eta, \eta| + e^{-im\pi} q_2 |\eta, \eta\rangle\langle-\eta, -\eta| + |-\eta, -\eta\rangle\langle-\eta, -\eta| \bigg] \tag{11}$$

where $q_2 = p^{n-2}$. The reduced two qubit density (11) can be alternatively written as

$$\rho_{12} = \frac{1}{2}\,(1+p^{n-2})\,\frac{\mathcal{N}^2}{\mathcal{N}_{2+}{}^2}\,|\eta\rangle_2\,{}_2\langle\eta| + \frac{1}{2}\,(1-p^{n-2})\,\frac{\mathcal{N}^2}{\mathcal{N}_{2-}{}^2}\,Z|\eta\rangle_2\,{}_2\langle\eta|Z \tag{12}$$

with

$$|\eta\rangle_2 = \mathcal{N}_{2+}(|\eta, \eta\rangle + e^{im\pi}|-\eta, -\eta\rangle) \quad \text{and} \quad Z|\eta\rangle_2 = \mathcal{N}_{2-}(|\eta, \eta\rangle - e^{im\pi}|-\eta, -\eta\rangle).$$

The normalization factors are defined by

$$\mathcal{N}_{2\pm}^{-2} = 2(1 \pm p^2 \cos m\pi).$$

In the computational base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the density matrix ρ_{12} has the form of the alphabet X. Indeed, it is represented by

$$\rho_{12} = 2\mathcal{N}^2 \begin{pmatrix} q_{2+}a_+^4 & 0 & 0 & q_{2+}a_+^2a_-^2 \\ 0 & q_{2-}a_+^2a_-^2 & q_{2-}a_+^2a_-^2 & 0 \\ 0 & q_{2-}a_+^2a_-^2 & q_{2-}a_+^2a_-^2 & 0 \\ q_{2+}a_+^2a_-^2 & 0 & 0 & q_{2+}a_-^4 \end{pmatrix} \tag{13}$$

where

$$a_\pm = \frac{\sqrt{1 \pm p}}{\sqrt{2}} \qquad \text{and} \qquad q_{2\pm} = 1 \pm q_2 \cos m\pi.$$

The state ρ_{12} can be written also as

$$\rho_{12} = \sum_{k,l=0,1} \rho^{kl} \otimes |k\rangle\langle l|. \tag{14}$$

This form is appropriate to establish a relation between the Bloch components of the 2×2 matrices ρ^{kl} and the correlation matrix elements associated with the two-qubit state ρ_{12}. In equation (14), the matrices ρ^{kl} writes in Bloch representation as

$$\rho^{00} = \frac{1}{2}(T_0^{00}\sigma_0 + T_3^{00}\sigma_3) \qquad \rho^{11} = \frac{1}{2}(T_0^{11}\sigma_0 + T_3^{11}\sigma_3) \tag{15}$$

and

$$\rho^{01} = \frac{1}{2}(T_1^{01}\sigma_1 + T_2^{01}\sigma_2) \qquad \rho^{10} = \frac{1}{2}(T_1^{10}\sigma_1 + T_2^{10}\sigma_2) \tag{16}$$

where the Bloch components T_α^{kl} $(\alpha = 0, 1, 2, 3)$ are

$$T_0^{kk} = \mathcal{N}^2(1 + (-)^k p)(1 + (-)^k p^{n-1}\cos m\pi), \quad T_3^{kk} = \mathcal{N}^2(1 + (-)^k p)(1 + (-)^k p^{n-2}\cos m\pi)$$

for $k = 0, 1$, and

$$T_1^{01} = T_1^{10} = \mathcal{N}^2(1 - p^2), \qquad T_2^{01} = -T_2^{10} = i\mathcal{N}^2(1 - p^2)p^{n-2}\cos m\pi.$$

Reporting (15) and (16) in (14), one gets

$$\rho_{12} = \sum_{\alpha\beta} T_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta \tag{17}$$

where the non vanishing matrix elements $T_{\alpha\beta}$ $(\alpha, \beta = 0, 1, 2, 3)$ are given by

$$T_{\alpha 0} = T_\alpha^{00} + T_\alpha^{11} \;\; \text{for } \alpha = 0, 3 \qquad T_{\alpha 1} = T_\alpha^{01} + T_\alpha^{10} \;\; \text{for } \alpha = 1$$

$$T_{\alpha 2} = iT_\alpha^{01} - iT_\alpha^{10} \;\; \text{for } \alpha = 2 \qquad T_{\alpha 3} = T_\alpha^{00} - T_\alpha^{11} \;\; \text{for } \alpha = 0, 3 \tag{18}$$

which gives

$$T_{00} = 1, \quad T_{11} = 2\mathcal{N}^2(1 - p^2), \quad T_{22} = -2\mathcal{N}^2(1 - p^2)\; p^{n-2}\cos m\pi,$$

$$T_{33} = 2\mathcal{N}^2(p^2 + p^{n-2}\cos m\pi), \quad T_{03} = T_{30} = 2\mathcal{N}^2(p + p^{n-1}\cos m\pi). \tag{19}$$

The expressions (18) establish the relations between the Bloch components T_α^{kl} associated with one qubit states (15) and (16) and the two qubit Fano-Bloch tensor elements occurring in the two qubit density ρ_{12} (17). This result is ready generalizable to subsystems comprising three or more qubits. This issue is discussed in what follows.

3.2 Three-qubit states

The three-qubit states is extracted from the whole state (6) by removing $(n - 3)$ qubits by the usual trace procedure. In this scenario, it is simple to check that all the three qubit density matrices are identical. Explicitly, they are given by

$$\rho_{123} = \mathcal{N}^2\Big[|\eta, \eta, \eta\rangle\langle\eta, \eta, \eta| + e^{im\pi}q_3| - \eta, -\eta, -\eta\rangle\langle\eta, \eta, \eta|$$

$$+ e^{-im\pi}q_3|\eta, \eta, \eta\rangle\langle -\eta, -\eta, -\eta| + | - \eta, -\eta, -\eta\rangle\langle -\eta, -\eta, -\eta|\Big] \tag{20}$$

where $q_3 = p^{n-3}$. It is simply verified that the mixed three qubit state ρ_{123} is of rank two and writes

$$\rho_{123} = \frac{1}{2}\,(1+p^{n-3})\,\frac{\mathcal{N}^2}{\mathcal{N}_{3+}^2}\,|\eta\rangle_3\,{}_3\langle\eta| + \frac{1}{2}\,(1-p^{n-3})\,\frac{\mathcal{N}^2}{\mathcal{N}_{3-}^2}\,Z|\eta\rangle_3\,{}_3\langle\eta|Z \tag{21}$$

where

$$|\eta\rangle_3 = \mathcal{N}_{3+}(|\eta,\eta,\eta\rangle + e^{im\pi}|-\eta,-\eta,-\eta\rangle) \qquad Z|\eta\rangle_3 = \mathcal{N}_{3-}(|\eta,\eta,\eta\rangle - e^{im\pi}|-\eta,-\eta,-\eta\rangle)$$

with the normalization factors $\mathcal{N}_{3\pm}$ given by

$$\mathcal{N}_{3\pm}^{-2} = 2(1 \pm p^3\cos m\pi).$$

In the standard computational basis $\{|000\rangle, |010\rangle, |100\rangle, |110\rangle, |001\rangle, |011\rangle, |101\rangle, |111\rangle\}$, the state ρ_{123} takes the matrix form

$$\frac{\rho_{123}}{2\mathcal{N}^2} = \begin{pmatrix}
q_{+3}a_+^6 & 0 & 0 & q_{+3}a_+^4a_-^2 & 0 & q_{+3}a_+^4a_-^2 & q_{+3}a_+^4a_-^2 & 0 \\
0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 & q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\
0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 & q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\
q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 & 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\
0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 & q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\
q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 & 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\
q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 & 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\
0 & q_{-3}a_+^2a_-^4 & q_{-3}a_+^2a_-^4 & 0 & q_{-3}a_+^2a_-^4 & 0 & 0 & q_{-3}a_-^6
\end{pmatrix}. \tag{22}$$

The density matrix is built of four blocks. The diagonal blocks appear as X alphabet with non-zero density matrix elements only along the diagonal and anti-diagonal contrarily to the two off diagonal blocks which have vanishing elements along the diagonal and anti-diagonal. This suggests that ρ_{123} can be viewed as a special instance of three qubit X states. Paralleling the treatment for two qubits, the state (22) is re-written as

$$\rho_{123} = \sum_{k,l=0,1} \rho^{kl} \otimes |k\rangle\langle l| \tag{23}$$

where $|k\rangle, |l\rangle$ belong to Hilbert space of the qubit 3. The two qubit density matrices ρ^{kk} ($k = 0, 1$), occurring in the expansion (23), writes in the computational basis spanned by $\{|0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2, |1\rangle_1 \otimes |0\rangle_2, |1\rangle_1 \otimes |1\rangle_2\}$, as

$$\rho^{00} = 2\mathcal{N}^2 \begin{pmatrix}
q_{+3}a_+^6 & 0 & 0 & q_{+3}a_+^4a_-^2 \\
0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 \\
0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 \\
q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4
\end{pmatrix}, \tag{24}$$

and

$$\rho^{11} = 2\mathcal{N}^2 \begin{pmatrix}
q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\
0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\
0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \\
q_{-3}a_+^2a_-^4 & 0 & 0 & q_{-3}a_-^6
\end{pmatrix}. \tag{25}$$

For $(k=0, l=1)$ and $(k=1, l=0)$, we have respectively

$$\rho^{01} = 2\mathcal{N}^2 \begin{pmatrix} 0 & q_{+3}a_+^4a_-^2 & q_{+3}a_+^4a_-^2 & 0 \\ q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\ q_{-3}a_+^4a_-^2 & 0 & 0 & q_{-3}a_+^2a_-^4 \\ 0 & q_{+3}a_+^2a_-^4 & q_{+3}a_+^2a_-^4 & 0 \end{pmatrix}, \tag{26}$$

and

$$\rho^{10} = 2\mathcal{N}^2 \begin{pmatrix} 0 & q_{-3}a_+^4a_-^2 & q_{-3}a_+^4a_-^2 & 0 \\ q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 \\ q_{+3}a_+^4a_-^2 & 0 & 0 & q_{+3}a_+^2a_-^4 \\ 0 & q_{-3}a_+^2a_-^4 & q_{-3}a_+^2a_-^4 & 0 \end{pmatrix}. \tag{27}$$

The Fano-Bloch representation of the matrices ρ^{kk}, given by (24) and (25), take the the form

$$\rho^{kk} = \frac{1}{4}\sum_{\alpha\beta} T^{kk}_{\alpha\beta}\ \sigma_\alpha \otimes \sigma_\beta \tag{28}$$

where $\alpha, \beta = 0,1,2,3$ and the correlation matrix elements $T^{kk}_{\alpha\beta}$ are given by

$$T^{kk}_{\alpha\beta} = \mathrm{Tr}(\rho^{kk}\ \sigma_\alpha \otimes \sigma_\beta).$$

The explicit expressions of the non vanishing correlation elements, in terms of the overlap factor p defined by (7), are

$$\begin{aligned} T^{kk}_{00} &= 1 \\ T^{kk}_{30} = T^{kk}_{03} &= \frac{p}{2}\ (1+(-)^k p)\ \frac{1+(-)^k p^{n-3}\cos m\pi}{1+p^n\cos m\pi} \\ T^{kk}_{11} &= \frac{1}{2}\ (1+(-)^k p)\ \frac{1-p^2}{1+p^n\cos m\pi} \\ T^{kk}_{22} &= -\frac{1}{2}\ (1+(-)^k p)\ \frac{(1-p^2)p^{n-3}\cos m\pi}{1+p^n\cos m\pi} \\ T^{kk}_{33} &= \frac{1}{2}\ (1+(-)^k p)\ \frac{p^2+(-)^k p^{n-3}\cos m\pi}{1+p^n\cos m\pi} \end{aligned} \tag{29}$$

Similarly, for the two-qubit states ρ^{kl} $(k \neq l)$ given by (26) and (27), the Fano-Bloch representation writes

$$\rho^{kl} = \frac{1}{4}\sum_{\alpha\beta} T^{kl}_{\alpha\beta}\ \sigma_\alpha \otimes \sigma_\beta \tag{30}$$

where the non zero matrix elements $T^{kl}_{\alpha\beta}$ are given by

$$T^{kl}_{01} = T^{kl}_{10} = \frac{1}{2}\ \frac{1-p^2}{1+p^n\cos m\pi}$$

$$T^{kl}_{02} = T^{kl}_{20} = (-)^k\frac{i}{2}\ \frac{p(1-p^2)}{1+p^n\cos m\pi}$$

$$T_{13}^{kl} = T_{31}^{kl} = \frac{1}{2} \; \frac{(1-p^2)p^{n-2}\cos m\pi)}{1+p^n\cos m\pi}$$

$$T_{23}^{kl} = T_{32}^{kl} = (-)^k \frac{i}{2} \; \frac{(1-p^2)p^{n-2}\cos m\pi)}{1+p^n\cos m\pi}. \tag{31}$$

Using (23), the three-qubit state ρ_{123} expands as

$$\rho_{123} = \frac{1}{2}\bigg[(\rho^{00}+\rho^{11})\otimes\sigma_0 + (\rho^{00}-\rho^{11})\otimes\sigma_3 + (\rho^{01}+\rho^{10})\otimes\sigma_1 + i(\rho^{01}-\rho^{10})\otimes\sigma_2\bigg]. \tag{32}$$

Inserting (28) and (30) in the expression (32) and using the results (29) and (31) , one gets

$$\rho_{123} = \frac{1}{8}\sum_{\alpha\beta}\bigg[T_{\alpha\beta 0}\;\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_0 + T_{\alpha\beta 1}\;\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_1 + T_{\alpha\beta 2}\;\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_2 + T_{\alpha\beta 3}\;\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_3\bigg] \tag{33}$$

where

$$T_{\alpha\beta 0} = T_{\alpha\beta}^{++} = T_{\alpha\beta}^{00} + T_{\alpha\beta}^{11}$$

$$T_{\alpha\beta 3} = T_{\alpha\beta}^{--} = T_{\alpha\beta}^{00} - T_{\alpha\beta}^{11} \tag{34}$$

with $\alpha\beta = 00, 03, 30, 11, 22, 33$ (cf. (29)), and

$$T_{\alpha\beta 1} = T_{\alpha\beta}^{+-} = T_{\alpha\beta}^{01} + T_{\alpha\beta}^{10}$$

$$T_{\alpha\beta 2} = T_{\alpha\beta}^{-+} = iT_{\alpha\beta}^{01} - iT_{\alpha\beta}^{10}. \tag{35}$$

with $\alpha\beta = 01, 02, 10, 20, 13, 23, 31, 32$ (cf. equations (31)). Reporting (29) and (31) in the expressions (34) and (35), one obtains the 32 non vanishing correlation matrix elements $T_{\alpha\beta\gamma}$ corresponding to the three qubit state ρ_{123}. Subsequently, The recursive relations (34) and (35) offer a nice tool to determine the correlation elements $T_{\alpha\beta\gamma}$ in terms of those associated with the two qubit density matrices ρ^{kk} and ρ^{kl} respectively given by (28) and (30). This recursive procedure is useful in simplifying the evaluation of pairwise correlations in three qubit state of type ρ_{123} as we shall discuss hereafter.

3.3 k-qubit states

The previous analysis can be extended to embrace an arbitrary mixed k-qubit state ($k = 2, 3, \cdots, n$) obtained by tracing out $(n-k)$ qubit from the state (6). The resulting reduced density is given by

$$\rho_{123\cdots k} = \mathcal{N}^2\bigg[|\eta,\eta,\cdots,\eta\rangle\langle\eta,\eta,\cdots,\eta| + e^{im\pi}q_k|-\eta,-\eta,\cdots,-\eta\rangle\langle\eta,\eta,\cdots,\eta|$$

$$+e^{-im\pi}q_k|\eta,\eta,\cdots,\eta\rangle\langle-\eta,-\eta,\cdots,-\eta| + |-\eta,-\eta,\cdots,-\eta\rangle\langle-\eta,-\eta,\cdots-\eta|\bigg] \tag{36}$$

where $q_k = p^{n-k}$. The reduced density matrix $\rho_{123\cdots k}$ is of rank 2. Indeed, the state (36) rewrites

$$\rho_{123\cdots k} = \frac{1}{2}\;(1+p^{n-k})\;\frac{\mathcal{N}^2}{\mathcal{N}_{k+}^2}\;|\eta\rangle_k\;{}_k\langle\eta| + \frac{1}{2}\;(1-p^{n-k})\;\frac{\mathcal{N}^2}{\mathcal{N}_{k-}^2}\;Z|\eta\rangle_k\;{}_k\langle\eta|Z \tag{37}$$

where

$$|\eta\rangle_k = \mathcal{N}_{k+}(|\eta,\eta,\cdots,\eta\rangle + e^{im\pi}|-\eta,-\eta,\cdots,-\eta\rangle) \qquad Z|\eta\rangle_k = \mathcal{N}_{k-}(|\eta,\eta,\cdots,\eta\rangle - e^{im\pi}|-\eta,-\eta,\cdots,-\eta\rangle)$$

and the normalization factors $\mathcal{N}_{k\pm}$ are given by

$$\mathcal{N}_{k\pm}^{-2} = 2(1 \pm p^k \cos m\pi).$$

The cyclic operator Z is now defined by

$$Z|\eta,\eta,\cdots,\eta\rangle = |\eta,\eta,\cdots,\eta\rangle \qquad Z|-\eta,-\eta,\cdots,-\eta\rangle = -|-\eta,-\eta,\cdots,-\eta\rangle.$$

Clearly, for $k=2$ and $k=3$, the state $\rho_{123\cdots k}$ (36) reduces to ρ_{12} (11) and ρ_{123} (20) respectively. Using (36), it is simple to check that the k-qubit state $\rho_{123\cdots k}$ can be expressed in terms of states comprising $(k-1)$-qubits. The state $\rho_{123\cdots k}$ (36) can be written also as

$$\rho_{123\cdots k} = \sum_{rs=1,2} \rho^{rs}_{12\cdots(k-1)} \otimes |r\rangle\langle s| \tag{38}$$

where

$$\rho^{rs}_{12\cdots(k-1)} \equiv \rho^{rs} = a_+^{2-r-s} a_-^{r+s} \Bigg[\frac{1}{2}\,(1+p^{n-k})\,\frac{\mathcal{N}^2}{\mathcal{N}^2_{(k-1)+}}\; Z^r|\eta\rangle_{(k-1)}\;{}_{(k-1)}\langle\eta|Z^s$$
$$+\frac{1}{2}\,(1-p^{n-k})\,\frac{\mathcal{N}^2}{\mathcal{N}^2_{(k-1)-}}\; Z^{r+1}|\eta\rangle_{(k-1)}\;{}_{(k-1)}\langle\eta|Z^{s+1}\Bigg]. \tag{39}$$

Explicitly, the k-qubit matrix (38) writes

$$\rho_{123\cdots k} = \frac{1}{2}(\rho^{00}+\rho^{11})\otimes\sigma_0 + \frac{1}{2}(\rho^{01}+\rho^{10})\otimes\sigma_1 + \frac{i}{2}(\rho^{01}-\rho^{10})\otimes\sigma_2 + \frac{1}{2}(\rho^{00}-\rho^{11})\otimes\sigma_3 \tag{40}$$

and the $(k-1)$-qubit states $\rho^{rs} \equiv \rho^{rs}_{12\cdots(k-1)}$ can be expanded, in Fano-Bloch representation, as

$$\rho^{rs} = \frac{1}{2^{k-1}} \sum_{\alpha_1,\alpha_2,\cdots,\alpha_{k-1}} T^{rs}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} \sigma_{\alpha_1}\otimes\sigma_{\alpha_2}\otimes\cdots\otimes\sigma_{\alpha_{k-1}}. \tag{41}$$

Hence, reporting (41) in (38), the k-qubit state $\rho_{123\cdots k}$ takes the form

$$\rho_{123\cdots k} = \frac{1}{2^k} \sum_{\alpha_1,\alpha_2,\cdots,\alpha_{k-1},\alpha_k} T_{\alpha_1\alpha_2\cdots\alpha_{k-1}\alpha_k} \sigma_{\alpha_1}\otimes\sigma_{\alpha_2}\otimes\cdots\otimes\sigma_{\alpha_{k-1}}\otimes\sigma_{\alpha_k}. \tag{42}$$

where the correlation matrix elements $T_{\alpha_1\alpha_2\cdots\alpha_{k-1}\alpha_k}$ express in terms of the correlations coefficients occurring in (41) as

$$T_{\alpha_1\alpha_2\cdots\alpha_{k-1}0} = T^{00}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} + T^{11}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}$$
$$T_{\alpha_1\alpha_2\cdots\alpha_{k-1}3} = T^{00}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} - T^{11}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}$$
$$T_{\alpha_1\alpha_2\cdots\alpha_{k-1}1} = T^{01}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} + T^{10}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}$$
$$T_{\alpha_1\alpha_2\cdots\alpha_{k-1}2} = iT^{01}_{\alpha_1\alpha_2\cdots\alpha_{k-1}} - iT^{10}_{\alpha_1\alpha_2\cdots\alpha_{k-1}}. \tag{43}$$

This establishes the relations between the correlation matrix elements of k and $(k-1)$-qubit states. In this picture the correlation matrix elements associated with a k-qubit state can be recursively expressed in terms of ones involving two qubits. The recurrence relations (43) reduce to (18) for $k=2$ and to (34-35) for $k=3$. To illustrate further the algorithm in deriving relations of type (43), we consider the case of four qubits ($k=4$). In this situation, the density matrix (36) becomes

$$\rho_{1234} = \mathcal{N}^2 \bigg[|\eta,\eta,\eta,\eta\rangle\langle\eta,\eta,\eta,\eta| + e^{im\pi} q_4 |-\eta,-\eta,-\eta,-\eta\rangle\langle\eta,\eta,\eta,\eta|$$

$$+ e^{-im\pi} q_4 |\eta,\eta,\eta,\eta\rangle\langle-\eta,-\eta,-\eta,-\eta| + |-\eta,-\eta,-\eta,-\eta\rangle\langle-\eta,-\eta,-\eta-\eta| \bigg] \tag{44}$$

and the expression (38) gives

$$\rho_{1234} = \rho_{123}^{00} \otimes |0\rangle\langle 0| + \rho_{123}^{01} \otimes |0\rangle\langle 1| + \rho_{123}^{10} \otimes |1\rangle\langle 0| + \rho_{123}^{11} \otimes |1\rangle\langle 1| \tag{45}$$

where the three-qubit states ρ_{123}^{00}, ρ_{123}^{01}, ρ_{123}^{10} and ρ_{123}^{11} are given in the usual computational basis as

$$\frac{\rho_{123}^{00}}{2\mathcal{N}^2} = \begin{pmatrix}
q_{+4}a_+^8 & 0 & 0 & q_{+4}a_+^6a_-^2 & 0 & q_{+4}a_+^6a_-^2 & q_{+4}a_+^6a_-^2 & 0 \\
0 & q_{-4}a_+^6a_-^2 & q_{-4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\
0 & q_{-4}a_+^6a_-^2 & q_{-4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\
q_{+4}a_+^6a_-^2 & 0 & 0 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\
0 & q_{-4}a_+^6a_-^2 & q_{-4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\
q_{+4}a_+^6a_-^2 & 0 & 0 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\
q_{+4}a_+^6a_-^2 & 0 & 0 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\
0 & q_{-4}a_+^4a_-^4 & q_{-4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6
\end{pmatrix} \tag{46}$$

$$\frac{\rho_{123}^{11}}{2\mathcal{N}^2} = \begin{pmatrix}
q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & q_{-4}a_+^4a_-^4 & 0 \\
0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\
0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\
q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \\
0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\
q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \\
q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0 \\
0 & q_{+4}a_+^2a_-^6 & q_{+4}a_+^2a_-^6 & 0 & q_{+4}a_+^2a_-^6 & 0 & 0 & q_{+4}a_-^8
\end{pmatrix}, \tag{47}$$

$$\frac{\rho_{123}^{01}}{2\mathcal{N}^2} = \begin{pmatrix}
0 & q_{+4}a_+^6a_-^2 & q_{+4}a_+^6a_-^2 & 0 & q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 \\
q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\
q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 \\
0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 \\
q_{-4}a_+^6a_-^2 & 0 & 0 & q_{-4}a_+^4a_-^4 & 0 & q_{-4}a_+^4a_-^4 & q_{-4}a_+^4a_-^4 & 0 \\
0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\
0 & q_{+4}a_+^4a_-^4 & q_{+4}a_+^4a_-^4 & 0 & q_{+4}a_+^4a_-^4 & 0 & 0 & q_{+4}a_+^2a_-^6 \\
q_{-4}a_+^4a_-^4 & 0 & 0 & q_{-4}a_+^2a_-^6 & 0 & q_{-4}a_+^2a_-^6 & q_{-4}a_+^2a_-^6 & 0
\end{pmatrix}, \tag{48}$$

and

$$\rho_{123}^{10} = \left(\rho_{123}^{01}\right)^t. \tag{49}$$

Obviously, with increasing number of qubits, the size of density matrices is growing. This induces inevitably complicated algebraic manipulation in deriving computable expressions of quantum correlations. However, the recursive algorithm presented above, offers an alternative way for symmetric multi-qubit (6), to reduce the complexity in determining analytical evaluation of geometric discord. The expression (45) allows us to write the correlations factors $T_{\alpha_1\alpha_2\alpha_3\alpha_4}$ in terms of those corresponding to three-qubit density matrices ρ_{123}^{00}, ρ_{123}^{01}, ρ_{123}^{10} and ρ_{123}^{11} (cf. matrices (46)-(49)). Indeed, the state ρ_{1234} expands in the Fano-Bloch representation as

$$\rho_{1234} = \frac{1}{2^4}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} T_{\alpha_1\alpha_2\alpha_3\alpha_4}\sigma_{\alpha_1}\otimes\sigma_{\alpha_2}\otimes\sigma_{\alpha_3}\otimes\sigma_{\alpha_4} \tag{50}$$

and by re-equating (45) as

$$\rho_{1234} = \frac{1}{2}(\rho_{123}^{00}+\rho_{123}^{11})\otimes\sigma_0 + \frac{1}{2}(\rho_{123}^{01}+\rho_{123}^{10})\otimes\sigma_1 + \frac{i}{2}(\rho_{123}^{01}-\rho_{123}^{10})\otimes\sigma_2 + \frac{1}{2}(\rho_{123}^{00}-\rho_{123}^{00})\otimes\sigma_3, \tag{51}$$

it is simple to see that

$$\begin{aligned}
T_{\alpha_1\alpha_2\alpha_3 0} &= T^{00}_{\alpha_1\alpha_2\alpha_3} + T^{11}_{\alpha_1\alpha_2\alpha_3}\\
T_{\alpha_1\alpha_2\alpha_3 1} &= T^{01}_{\alpha_1\alpha_2\alpha_3} + T^{10}_{\alpha_1\alpha_2\alpha_3}\\
T_{\alpha_1\alpha_2\alpha_3 2} &= iT^{01}_{\alpha_1\alpha_2\alpha_3} - iT^{10}_{\alpha_1\alpha_2\alpha_3}\\
T_{\alpha_1\alpha_2\alpha_3 3} &= T^{00}_{\alpha_1\alpha_2\alpha_3} - T^{11}_{\alpha_1\alpha_2\alpha_3}
\end{aligned} \tag{52}$$

where the quantities $T^{kl}_{\alpha_1,\alpha_2,\alpha_3}$, defined so that

$$\rho_{123}^{kl} = \frac{1}{2^3}\sum_{\alpha_1,\alpha_2,\alpha_3} T^{kl}_{\alpha_1\alpha_2\alpha_3}\sigma_{\alpha_1}\otimes\sigma_{\alpha_2}\otimes\sigma_{\alpha_3}, \tag{53}$$

can be obtained easily following the method developed above for three and two qubit states (modulo some obvious substitutions). In this manner, one verifies that the only non vanishing elements $T_{\alpha_1\alpha_2\alpha_3\alpha_4}$ are those with indices $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ belonging to the following set of quadruples

$$\begin{gathered}
\{\{00,11,22,33,03,30\}\times\{0,3\}\times\{0,3\},\\
\{00,11,22,33,03,30\}\times\{1,2\}\times\{1,2\},\\
\{01,10,20,02,13,31,23,32\}\times\{1,2\}\times\{0,3\},\\
\{01,10,20,02,13,31,23,32\}\times\{0,3\}\times\{1,2\}\}.
\end{gathered}$$

Finally, we stress that the recursive approach giving the Fano-Bloch components for an arbitrary k-qubit state in terms of those involving $(k-1)$-qubits is of central importance for our purpose. This gives a simple way to specify the correlation matrix elements for k-qubits state in terms of ones associated with its sub-components which render more tractable the computation of all pairwise geometric quantum discord in the multi-qubit Shroödinger cat states (6).

4 Geometric measure of quantum discord and classical states

We now face the question of determining the explicit form of pairwise geometric discord in the k-qubit mixed state (36) when partitioned into two subsystems: one containing a single qubit and the second part is grouping the rest of the $(k-1)$ qubits living in 2^{k-1} dimensional Hilbert space. For this end, it is necessary to find the expression of closest classical states to the states of type (36) when the distance is measured by Hilbert-Schmidt norm defined by

$$||\rho - \rho'||^2 = \mathrm{Tr}\big[(\rho - \rho')^2\big]$$

for all density operators acting on the Hilbert space $\mathcal{H}_n$. To extend the procedure developed in [15], we shall begin with the simplest case $k = 2$. A special focus concerns the derivation of the closest classical states to the two-qubit state (17).

4.1 Two-qubit states

For the density matrix (17), which reads also as

$$\rho_{12} = \frac{1}{4}\bigg[\sigma_0 \otimes \sigma_0 + T_{30}\ \sigma_3 \otimes \sigma_0 + T_{03}\ \sigma_0 \otimes \sigma_3 + T_{11}\ \sigma_1 \otimes \sigma_1 + T_{22}\ \sigma_2 \otimes \sigma_2 + T_{33}\ \sigma_3 \otimes \sigma_3\bigg], \qquad (54)$$

the zero-discord or classical states are given by

$$\chi_{12} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^2 + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^2 \qquad (55)$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit 1 and ρ_i^2 $(i = 1, 2)$ are reduced density matrices attached the second qubit. It can be rewritten as

$$\chi_{12} = \frac{1}{4}\bigg[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} te_i\ \ \sigma_i \otimes \sigma_0 + \sum_{i=1}^{3}(s_+)_i\ \sigma_0 \otimes \sigma_i + \sum_{i,j=1}^{3} e_i(s_-)_j\ \sigma_i \otimes \sigma_j\bigg] \qquad (56)$$

where

$$t = p_1 - p_2 \qquad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle \qquad (s_\pm)_j = \mathrm{Tr}\big((p_1\rho_1^2 \pm p_2\rho_2^2)\sigma_j\big).$$

The distance between the density matrix ρ_{12} (54) and the classical state χ_{12} (56), as measured by Hilbert-Schmidt norm, is

$$||\rho_{12} - \chi_{12}||^2 = \frac{1}{4}\bigg[(t^2 - 2te_3T_{30} + T_{30}^2) + \sum_{i=1}^{3}(T_{0i} - (s_+)_i)^2 + \sum_{i,j=1}^{3}(T_{ij} - e_i(s_-)_j)^2\bigg]. \qquad (57)$$

The minimal distance is obtained by minimizing the the Hilbert-Schmidt norm (57) with respect to the parameters t, $(s_+)_i$ and $(s_-)_i$. This gives

$$t = e_3T_{30}$$

$$(s_+)_1 = 0 \quad (s_+)_2 = 0 \quad (s_+)_3 = T_{03}$$

$$(s_-)_i = \sum_{j=1}^{3} e_j T_{ji}. \tag{58}$$

Inserting the solutions (58) in (57), one gets

$$||\rho_{12} - \chi_{12}||^2 = \frac{1}{4}\bigg[\mathrm{Tr}K - \vec{e}^{\,t}\ K\ \vec{e}\bigg] \tag{59}$$

where $\vec{e}^{\,t} = (e_1, e_2, e_3)$ and the matrix K is defined by

$$K = \mathrm{diag}(T_{11}^2, T_{22}^2, T_{30}^2 + T_{33}^2). \tag{60}$$

From the result (19), the eigenvalues of the matrix K (60) read

$$k_1 \equiv T_{11}^2 = \frac{(1-p^2)^2}{(1+p^n \cos m\pi)^2}, \tag{61}$$

$$k_2 \equiv T_{22}^2 = \frac{(1-p^2)^2 p^{2(n-2)}}{(1+p^n \cos m\pi)^2}, \tag{62}$$

$$k_3 \equiv T_{30}^2 + T_{33}^2 = \frac{(p^2 + p^{2(n-2)})(1+p^2) + 4p^n \cos m\pi}{(1+p^n \cos m\pi)^2}. \tag{63}$$

Using (59), it is easily seen that the minimal Hilbert-Schmidt distance is obtained for the vector $\vec{e}$ associated with the maximal eigenvalue $k_{\max} = \max(k_1, k_2, k_3)$. Thus, the geometric measure of quantum discord in the state ρ_{12} is given by

$$D_{\mathrm{g}}(\rho_{12}) = \frac{1}{4}(k_1 + k_2 + k_3 - k_{\max}). \tag{64}$$

Comparing (61) and (62), we have $k_2 < k_1$. This implies that $k_{\max}$ can be either k_1 or k_3. In this respect, to find the closest classical states, two situations must be considered separately. We begin first with the case where $k_{\max} = k_3$. The eigenvector associated with this maximal eigenvalue is $\vec{e} = (e_1 = 0, e_2 = 0, e_3 = 1)^t$. Reporting this result in (58),the closest classical state (56) is simply given by

$$\chi_{12}^{(3)} = \frac{1}{4}\bigg[\sigma_0 \otimes \sigma_0 + T_{30}\ \ \sigma_3 \otimes \sigma_0 + T_{03}\ \sigma_0 \otimes \sigma_3 + T_{33}\ \sigma_3 \otimes \sigma_3\bigg]. \tag{65}$$

Similarly, the eigenvector associated to $k_{\max} = k_1$ is $\vec{e} = (e_1 = 1, e_2 = 0, e_3 = 0)^t$ and from (58) one gets

$$\chi_{12}^{(1)} = \frac{1}{4}\bigg[\sigma_0 \otimes \sigma_0 + T_{03}\ \sigma_0 \otimes \sigma_3 + T_{11}\ \sigma_1 \otimes \sigma_1\bigg]. \tag{66}$$

Beside the explicit derivation of closest classical states (65) and (66), another important point to be emphasized is the relation between the matrix K (60), which encodes the geometric measure quantum correlations in the state ρ_{12}, and the Bloch components of the one-qubit density matrices ρ^{ii} $(i = 1, 2)$ and ρ^{ij} $(i \neq j)$ given respectively by (15) and (16). For this purpose, we use the relations (18) to rewrite the matrix K (60) as

$$K = \mathrm{diag}(2(T_1^{01})^2, -2(T_2^{01})^2, (T_3^{00})^2 + (T_3^{11})^2). \tag{67}$$

Furthermore, for one-qubit states ρ^{00}, ρ^{01}, ρ^{10} and ρ^{11}, we introduce the analogues of the matrix K (60). For the states ρ^{00} and ρ^{11} (15), we define the 3×3 matrix

$$K^{kk} = (0, 0, T_3^{kk})^t \ (0, 0, T_3^{kk}), \qquad k = 0, 1,$$

and similarly, we introduce the matrices

$$K^{kl} = (T_1^{kl}, iT_2^{kl}, 0)^t \ (T_1^{kl}, iT_2^{kl}, 0) \quad \text{for} \quad (k, l) = (0, 1) \text{ or } (1, 0).$$

for the states ρ^{01} and ρ^{10} (16). They satisfied the remarkable additivity relation

$$K = 2(K^{00} + K^{01} + K^{10} + K^{11}). \tag{68}$$

This relation holds also for the states containing three or more qubits as a consequence of the symmetry invariance of the multi-qubit system under consideration.

4.2 Three-qubit states

We now extend the previous results to the three qubit states of the form (20). More especially, we analytically determine the pairwise quantum discord between the qubit 1 and the two qubit subsystem (23). We also give the explicit forms of the closest classical tripartite states. To begin, we write the density matrix (33) as follows

$$\rho_{123} = \frac{1}{8}\Big[T_{000}\ \sigma_0\otimes\sigma_0\otimes\sigma_0 + T_{300}\ \sigma_3\otimes\sigma_0\otimes\sigma_0 + \sum_{(\beta,\gamma)\neq(0,0)} T_{0\beta\gamma}\ \sigma_0\otimes\sigma_\beta\otimes\sigma_\gamma + \sum_i \sum_{(\beta,\gamma)\neq(0,0)} T_{i\beta\gamma}\ \sigma_i\otimes\sigma_\beta\otimes\sigma_\gamma\Big]. \tag{69}$$

The classical states (i.e., states presenting zero discord between the qubit 1 and the subsystem 23) are of the form

$$\chi_{1|23} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^{23} + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^{23} \tag{70}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit 1. The density matrices ρ_i^{23} $(i = 1, 2)$ corresponding to the subsystem 23 write as

$$\rho_i^{23} = \frac{1}{4}\Big[\sum_{\alpha,\beta} \mathrm{Tr}(\rho_i^{23}\sigma_\alpha \otimes \sigma_\beta)\sigma_\alpha \otimes \sigma_\beta\Big].$$

The Fano-Bloch form of the tripartite classical state (70) is given by

$$\chi_{1|23} = \frac{1}{8}\Big[\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} te_i \quad \sigma_i \otimes \sigma_0 \otimes \sigma_0$$

$$+ \sum_{(\alpha,\beta)\neq(0,0)} (s_+)_{\alpha,\beta}\ \sigma_0 \otimes \sigma_\alpha \otimes \sigma_\beta + \sum_{i=1}^{3} \sum_{(\alpha,\beta)\neq(0,0)} e_i(s_-)_{\alpha,\beta}\ \sigma_i \otimes \sigma_\alpha \otimes \sigma_\beta\Big] \tag{71}$$

where

$$t = p_1 - p_2 \qquad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle \qquad (s_\pm)_{\alpha,\beta} = \mathrm{Tr}\big((p_1\rho_1^{23} \pm p_2\rho_2^{23})\sigma_\alpha \otimes \sigma_\beta\big).$$

The Hilbert-Schmidt distance between the three qubit state ρ_{123} (69) and a classical state, having the form (71), is

$$||\rho_{1|23}-\chi_{1|23}||^2=\frac{1}{8}\bigg[(t^2-2te_3T_{300}+T_{300}^2)+\sum_{(\alpha,\beta)\neq(0,0)}(T_{0\alpha\beta}-(s_+)_{\alpha,\beta})^2+\sum_{i=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)}(T_{i\alpha\beta}-e_i(s_-)_{\alpha,\beta})^2\bigg]. \tag{72}$$

To derive the closest classical state as measured by Hilbert-Schmidt, an optimization with respect to the parameters t, e_i ($i=1,2,3$) and $(s_\pm)_{\alpha,\beta}$ is performed. The minimal distance is attainable by setting zero the partial derivatives of the Hilbert-Schmidt distance (72) with respect to t and $(s_\pm)_{\alpha,\beta}$. This gives

$$t=e_3T_{300} \qquad (s_+)_{\alpha,\beta}=T_{0\alpha\beta} \qquad (s_-)_{\alpha,\beta}=\sum_{i=1}^{3}e_iT_{i\alpha\beta}. \tag{73}$$

Reporting the solutions (73) in (72), one obtains

$$||\rho_{1|23}-\chi_{1|23}||^2=\frac{1}{8}\bigg[T_{300}^2-e_3^2T_{300}^2+\sum_{i=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)}T_{i\alpha\beta}^2-\sum_{i,j=1}^{3}\sum_{(\alpha,\beta)\neq(0,0)}e_ie_jT_{i\alpha\beta}T_{j\alpha\beta}\bigg] \tag{74}$$

which has to be optimized with respect to the three components of the unit vector $\vec{e}=(e_1,e_2,e_3)$. Inspired by the two qubit example, a more convenient form of the distance (74) is

$$||\rho_{1|23}-\chi_{1|23}||^2=\frac{1}{8}\big[||x||^2+||T||^2-\vec{e}(xx^t+TT^t)\vec{e}^{\,t}\big] \tag{75}$$

in terms of the 1×3 matrix defined by

$$x^t:=(0,0,T_{300}) \tag{76}$$

and the 3×15 matrix given by

$$T:=(T_{i\alpha\beta}) \quad \text{with} \quad (\alpha,\beta)\neq(0,0). \tag{77}$$

Setting

$$K=xx^t+TT^t, \tag{78}$$

and reporting (76) and (77) in (78), one obtains after some tedious calculations

$$K=\mathrm{diag}(k_1,k_2,k_3) \tag{79}$$

where k_1, k_2 and k_3 are given by

$$k_1=\sum_{i=1,2}\sum_{j=0,3}T_{1ij}^2+T_{1ji}^2, \quad k_2=\sum_{i=1,2}\sum_{j=0,3}T_{2ij}^2+T_{2ji}^2, \quad k_3=\sum_{i=0,3}\sum_{j=0,3}T_{3ij}^2+\sum_{i=1,2}\sum_{j=1,2}T_{3ij}^2. \tag{80}$$

Using the relations (34) and (35), the eigenvalues of the matrix K can be re-expressed in terms of the bipartite correlations elements $T_{\alpha\beta}$ associated with the the qubit density matrices ρ^{00}, ρ^{01}, ρ^{10} and ρ^{11} (cf. (28) and (30)). Therefore, one has

$$k_1=2[(T_{11}^{00})^2+(T_{11}^{11})^2]+4|T_{10}^{01}|^2+4|T_{13}^{01}|^2, \tag{81}$$

$$k_2 = 2[(T_{22}^{00})^2 + (T_{22}^{11})^2] + 4|T_{20}^{01}|^2 + 4|T_{23}^{01}|^2, \tag{82}$$

$$k_3 = 2[(T_{30}^{00})^2 + (T_{30}^{11})^2] + 2[(T_{33}^{00})^2 + (T_{33}^{11})^2] + 4|T_{31}^{01}|^2 + 4|T_{32}^{01}|^2. \tag{83}$$

Finally, using (29) and (31), we obtain

$$k_1 = 2\ \frac{(1-p^2)^2(1+p^2)}{(1+p^n \cos m\pi)^2}, \tag{84}$$

$$k_2 = 2\ \frac{(1-p^2)^2(1+p^2)p^{2(n-3)}}{(1+p^n \cos m\pi)^2}, \tag{85}$$

$$k_3 = 2\ \frac{(p^2+p^{2(n-3)})(1+p^4) + 4p^n \cos m\pi}{(1+p^n \cos m\pi)^2}. \tag{86}$$

The minimal value of the Hilbert-Schmidt distance (75) is reached when $\vec{e}$ is the eigenvector associated to the largest eigenvalue of the matrix defined by (78). We denote by $k_{\max}$ the largest eigenvalue among k_1, k_2 and k_3. Since $k_1 \geq k_2$, $k_{\max}$ is k_2 or k_3 depending on the number of qubits n and the overlap p. Notice that the sum of the eigenvalues k_1, k_2 and k_2 of the matrix K is exactly the sum of the Hilbert-Schmidt norm of the matrices x (76) and T (77) (*i.e.* $k_1+k_2+k_3 = ||x||^2+||T||^2$). Accordingly, the minimal Hilbert-Schmidt distance (75) writes as

$$D_{\rm g}(\rho_{1|23}) = \frac{1}{8}(k_1 + k_2 + k_3 - k_{\max}) \tag{87}$$

and gives the geometric measure of the pairwise quantum discord in the state ρ_{123} partitioned in the subsystems 1 and 23. When the matrix elements of the density matrix ρ_{123} (32) are such that $k_{\max} = k_1$, the closest classical state is given by

$$\chi_{1|23}^{(1)} = \frac{1}{8}\left[\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sum_{(\alpha,\beta)\neq(0,0)} T_{0\alpha\beta}\ \sigma_0 \otimes \sigma_\alpha \otimes \sigma_\beta + \sum_{(\alpha,\beta)\neq(0,0)} T_{1\alpha\beta}\ \sigma_1 \otimes \sigma_\alpha \otimes \sigma_\beta\right]. \tag{88}$$

Conversely, in the situation where $k_{\max} = k_3$, one finds

$$\chi_{1|23}^{(3)} = \frac{1}{8}\Bigg[\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + T_{300}\ \sigma_3 \otimes \sigma_0 \otimes \sigma_0$$

$$+ \sum_{(\alpha,\beta)\neq(0,0)} T_{0\alpha\beta}\ \sigma_0 \otimes \sigma_\alpha \otimes \sigma_\beta + \sum_{(\alpha,\beta)\neq(0,0)} T_{3\alpha\beta}\ \sigma_3 \otimes \sigma_\alpha \otimes \sigma_\beta\Bigg]. \tag{89}$$

4.3 k-qubit states

Based on the previous analysis, we now determine the analytic expression of the geometric discord in the k-qubit state (36) when a bipartite splitting of type $1|23\cdots k$ is considered. In this respect, we expand the density matrix $\rho_{12\cdots k}$ (42) as

$$\rho_{12\cdots k} = \frac{1}{2^k}\Bigg[T_{00\cdots 0}\ \sigma_0 \otimes \sigma_0 \cdots \otimes \sigma_0 + T_{30\cdots 0}\ \sigma_3 \otimes \sigma_0 \otimes \cdots \sigma_0$$

$$+ \sum_{(\alpha_2,\cdots\alpha_k)\neq(0,\cdots,0)} T_{0\alpha_2\cdots\alpha_k}\ \sigma_0 \otimes \sigma_{\alpha_2} \otimes \cdots \sigma_{\alpha_k} + \sum_i \sum_{(\alpha_2,\cdots\alpha_k)\neq(0,\cdots,0)} T_{i\alpha_2\cdots\alpha_k}\ \sigma_i \otimes \sigma_{\alpha_2} \otimes \cdots \sigma_{\alpha_k}\Bigg] \tag{90}$$

in terms of the non vanishing correlations coefficients $T_{\alpha_1\alpha_2\cdots\alpha_k}$. Any k-qubit state having zero discord is necessarily of the form

$$\chi_{1|23\cdots k} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^{23\cdots k} + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^{23\cdots k} \tag{91}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit 1. The density matrices $\rho_i^{23\cdots k}$ $(i = 1, 2)$, corresponding to the subsystem $(23\cdots k)$ composed of $(k-1)$ qubits, write as

$$\rho_i^{23\cdots k} = \frac{1}{2^{k-1}}\bigg[\sum_{\alpha_2,\cdots\alpha_k} \mathrm{Tr}(\rho_i^{23\cdots k}\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k})\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k}\bigg].$$

To examine the pairwise quantum correlations in the states (36), the appropriate form for the classical state (91) is

$$\chi_{1|23\cdots k} = \frac{1}{2^k}\bigg[\sigma_0\otimes\sigma_0\cdots\otimes\sigma_0 + \sum_{i=1}^{3} te_i \quad \sigma_i\otimes\sigma_0\cdots\otimes\sigma_0$$

$$+ \sum_{(\alpha_2,\cdots\alpha_k)\neq(0,\cdots,0)} (s_+)_{\alpha_2,\cdots,\alpha_k}\ \sigma_0\otimes\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k} + \sum_{i=1}^{3}\sum_{(\alpha_2,\cdots\alpha_k)\neq(0,\cdots,0)} e_i(s_-)_{\alpha_2,\cdots,\alpha_k}\ \sigma_i\otimes\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k}\bigg] \tag{92}$$

where

$$t = p_1 - p_2 \qquad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle \qquad (s_\pm)_{\alpha_2,\cdots,\alpha_k} = \mathrm{Tr}\big((p_1\rho_1^{23\cdots k} \pm p_2\rho_2^{23\cdots k})\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_k}\big).$$

The Hilbert-Schmidt distance between the state $\rho_{123\cdots k}$ (90) and a classical state of the from (92) is given by the following expression

$$||\rho_{1|23\cdots k} - \chi_{1|23\cdots k}||^2 = \frac{1}{2^k}\bigg[(t^2 - 2te_3T_{30\cdots 0} + T_{30\cdots 0}^2) + \sum_{(\alpha_2,\cdots\alpha_k)\neq(0,\cdots,0)} (T_{0\alpha_2\cdots\alpha_k} - (s_+)_{\alpha_2,\cdots,\alpha_k})^2$$

$$+\sum_{i=1}^{3}\sum_{(\alpha_2,\cdots\alpha_k)\neq(0,\cdots,0)} (T_{i\alpha_2\cdots\alpha_k} - e_i(s_-)_{\alpha_2,\cdots,\alpha_k})^2\bigg] \tag{93}$$

which must be minimized with respect to the parameters t, e_i $(i = 1, 2, 3)$ and $(s_\pm)_{\alpha_2,\cdots,\alpha_k}$. Setting the partial derivatives of (93), with respect to the parameters t and $(s_\pm)_{\alpha_2,\cdots,\alpha_k}$, equal to zero, we get

$$t = e_3T_{30\cdots 0} \qquad (s_+)_{\alpha_2,\cdots,\alpha_k} = T_{0\alpha_2\cdots\alpha_k} \qquad (s_-)_{\alpha_2,\cdots,\alpha_k} = \sum_{i=1}^{3} e_iT_{i\alpha_2\cdots\alpha_k}. \tag{94}$$

In particular, these solutions reduce to equations (58) and (73) for $k = 2$ and $k = 3$ respectively. Reporting the conditions (94) in the expression (93), one obtains

$$||\rho_{1|23\cdots k} - \chi_{1|23\cdots k}||^2 = \frac{1}{2^k}\bigg[T_{30\cdots 0}^2 - e_3^2T_{30\cdots 0}^2 + \sum_{i=1}^{3}\sum_{(\alpha_2,\cdots,\alpha_k)\neq(0,0)} T_{i\alpha_2\cdots\alpha_k}^2$$

$$-\sum_{i,j=1}^{3}\sum_{(\alpha_2,\cdots,\alpha_k)\neq(0,\cdots,0)} e_ie_jT_{i\alpha_2\cdots\alpha_k}T_{j\alpha_2\cdots\alpha_k}\bigg]. \tag{95}$$

After some algebra, the distance (95) takes the following compact form

$$||\rho_{1|23\cdots k} - \chi_{1|23\cdots k}||^2 = \frac{1}{2^k}\left[||x||^2 + ||T||^2 - \vec{e}(xx^t + TT^t)\vec{e}^{\ t}\right] \tag{96}$$

in terms of the 1×3 matrix defined by

$$x^t = (0, 0, T_{30\cdots 0}) \tag{97}$$

and the $3 \times (4^{k-1} - 1)$ matrix given by

$$T = (T_{i\alpha_2\cdots\alpha_k}) \quad \text{with} \quad (\alpha_2, \cdots, \alpha_k) \neq (0, \cdots, 0) \tag{98}$$

which are the extended versions of the matrices (76) and (77) introduced for $k = 3$. The equation (96) suggests that the pairwise quantum correlation is completely characterized by the eigenvalues of the matrix

$$K = xx^t + TT^t. \tag{99}$$

The analytical expression of these eigenvalues for an arbitrary multi-qubit state constitutes a very complex task. However, this complexity is considerably reduced for the states $\rho_{12\cdots k}$ by exploiting their parity symmetry (i.e. commutes with $\sigma_3 \otimes \sigma_3 \cdots \sigma_3$). This implies that the matrix T (98) writes formally as

$$T = \sum_{\alpha_2,\cdots,\alpha_k} (T_{1\alpha_2\cdots\alpha_k}, T_{2\alpha_2\cdots\alpha_k}, 0)^t + \sum_{\alpha_2,\cdots,\alpha_k} (0, 0, T_{3\alpha_2\cdots\alpha_k})^t.$$

This form is more appropriate to show that the product TT^t is diagonal. The density matrix $\rho_{12\cdots k}$ possesses also exchange symmetry. Subsequently, since its elements are all reals and in view of the recurrence relation (43), the off-diagonal entries of the matrix TT^t vanish. This result has been shown already for $k = 2$ and $k = 3$ and will be explicitly proved hereafter for $k = 4$. It follows that the matrix K (99) is diagonal

$$K = \mathrm{diag}(k_1, k_2, k_3)$$

and the corresponding eigenvalues are

$$k_1 = \sum_{\alpha_2,\cdots,\alpha_k} T^2_{1\alpha_2\cdots\alpha_k}$$

$$k_2 = \sum_{\alpha_2,\cdots,\alpha_k} T^2_{2\alpha_2\cdots\alpha_k}$$

$$k_3 = T^2_{30\cdots 0} + \sum_{\alpha_2,\cdots,\alpha_k \neq 0} T^2_{3\alpha_2\cdots\alpha_k}.$$

They can be explicitly expressed in terms of the overlap p (7) like the special cases $k = 2$ and $k = 3$. This requires the expressions of k-components tensor correlations $T_{\alpha_1\alpha_2\cdots\alpha_k}$ in terms of those involving two qubits. This recursive procedure requires tedious but feasible calculations. To exemplify this method, we consider the situation where $k = 4$. In this case, the 3×63 matrix elements of T defined by (98) can be explicitly derived using the equation (52). A straightforward but lengthy computation shows that the 3×3 matrix K is diagonal and the corresponding eigenvalues are

$$\begin{aligned} k_1 = & \sum_{k=0,3}\sum_{i=1,2}\sum_{j=0,3} T^2_{1kji} + \sum_{k=1,2}\sum_{j=1,2}\sum_{i=1,2} T^2_{1kji} \\ & + \sum_{k=0,3}\sum_{j=1,2}\sum_{i=0,3} T^2_{1kji} + \sum_{k=1,2}\sum_{j=0,3}\sum_{i=0,3} T^2_{1kji}, \end{aligned} \tag{100}$$

$$\begin{aligned} k_2 = & \sum_{k=0,3}\sum_{i=1,2}\sum_{j=0,3} T^2_{2kji} + \sum_{k=1,2}\sum_{j=1,2}\sum_{i=1,2} T^2_{2kji} \\ & + \sum_{k=0,3}\sum_{j=1,2}\sum_{i=0,3} T^2_{2kji} + \sum_{k=1,2}\sum_{j=0,3}\sum_{i=0,3} T^2_{2kji}, \end{aligned} \tag{101}$$

$$\begin{aligned} k_3 = & \sum_{k=0,3}\sum_{i=0,3}\sum_{j=0,3} T^2_{3kji} + \sum_{k=1,2}\sum_{j=1,2}\sum_{i=0,3} T^2_{3kji} \\ & + \sum_{k=0,3}\sum_{j=1,2}\sum_{i=1,2} T^2_{3kji} + \sum_{k=1,2}\sum_{j=0,3}\sum_{i=1,2} T^2_{3kji}. \end{aligned} \tag{102}$$

The expressions (100), (101) and (102) can be simplified further. Indeed, from the relations (43), which reproduce the expressions (52) for $k = 4$, one obtains

$$\begin{aligned} k_1 = 2\Big[& \sum_{k=0,3}\sum_{j=1,2} ((T^{00}_{1kj})^2 + (T^{11}_{1jk})^2) + \sum_{k=1,2}\sum_{j=0,3} ((T^{00}_{1kj})^2 + (T^{11}_{1jk})^2)\Big] \\ & +4\Big[\sum_{k=0,3}\sum_{j=0,3} T^{01}_{1kj}T^{10}_{1kj} + \sum_{k=1,2}\sum_{j=1,2} T^{01}_{1kj}T^{10}_{1kj}\Big], \end{aligned} \tag{103}$$

$$\begin{aligned} k_2 = 2\Big[& \sum_{k=0,3}\sum_{j=1,2} ((T^{00}_{2kj})^2 + (T^{11}_{2jk})^2) + \sum_{k=1,2}\sum_{j=0,3} ((T^{00}_{2kj})^2 + (T^{11}_{2jk})^2)\Big] \\ & +4\Big[\sum_{k=0,3}\sum_{j=0,3} T^{01}_{2kj}T^{10}_{2kj} + \sum_{k=1,2}\sum_{j=1,2} T^{01}_{2kj}T^{10}_{1kj}\Big], \end{aligned} \tag{104}$$

$$\begin{aligned} k_3 = 2\Big[& \sum_{k=0,3}\sum_{j=0,3} ((T^{00}_{3kj})^2 + (T^{11}_{3jk})^2) + \sum_{k=1,2}\sum_{j=1,2} ((T^{00}_{3kj})^2 + (T^{11}_{3jk})^2)\Big] \\ & +4\Big[\sum_{k=0,3}\sum_{j=1,2} T^{01}_{3kj}T^{10}_{3kj} + \sum_{k=1,2}\sum_{j=0,3} T^{01}_{3kj}T^{10}_{3kj}\Big], \end{aligned} \tag{105}$$

in terms of the three qubit correlation elements $T^{kl}_{\alpha\beta\gamma}$ associated with the density matrices ρ^{kl}_{123} (53). The second step consists in expressing the 3-qubit correlation matrix elements $T^{kl}_{\alpha\beta\gamma}$ as expansion of bipartite correlations associated with two qubit subsystems $T^{kl}_{\alpha\beta}$. This is simply achieved by using the recurrence relations of type (34) and (35) (modulo some obvious substitution). Finally, one has

$$k_1 = 16\mathcal{N}^4(1-p^2)(1-p^6), \tag{106}$$

$$k_2 = 16\mathcal{N}^4(1-p^2)(1-p^6)p^{2(n-4)}, \tag{107}$$

$$k_3 = 16\mathcal{N}^4\Big[(1+p^6)(p^2+p^{2(n-4)}) + 4p^n\cos m\pi\Big], \tag{108}$$

and the pairwise quantum discord is

$$D_g(\rho_{1|234}) = \frac{1}{4}\,\min\{k_1+k_2, k_1+k_3, k_2+k_3\}. \tag{109}$$

This example illustrates that this hierarchical or recursive method, when carried out properly, leads to the pairwise quantum correlation between a single qubit and a $(k-1)$-qubits cluster in the k-qubit mixed states $\rho_{1|23\cdots k}$. But one should recognize that it involves lengthy expressions and tedious calculations.

5 Pairwise encoding

Another adequate approach to tackle the pairwise correlation in states of type $\rho_{1|23\cdots k}$, consists in mapping the physical states of the subsystem containing $(k-1)$-qubits onto two logical qubits. This idea was recently considered in [36, 43, 44] to examine the pairwise quantum correlations in multi-qubit systems. So in this section, we shall consider the scenario where the information contained in the cluster of $(23\cdots k)$ is encoded in two logical qubits $\{|0\rangle_{23\cdots k}, |1\rangle_{23\cdots k}\}$ defined by

$$|\eta,\eta,\cdots\eta\rangle \equiv b_+|0\rangle_{23\cdots k} + b_-|1\rangle_{23\cdots k} \qquad |-\eta,-\eta,\cdots,-\eta\rangle \equiv b_+|0\rangle_{23\cdots k} - b_-|1\rangle_{23\cdots k}\ , \tag{110}$$

where

$$b_\pm = \sqrt{\frac{1\pm p^{k-1}}{2}}.$$

According to this splitting scheme, the density matrix (36) rewrites, in the basis $\{|0\rangle\otimes|0\rangle_{23\cdots k}, |0\rangle\otimes|1\rangle_{23\cdots k}, |1\rangle\otimes|0\rangle_{23\cdots k}, |1\rangle\otimes|1\rangle_{23\cdots k}\}$, as

$$\rho_{1(23\cdots k)} = 2\mathcal{N}^2\begin{pmatrix} a_+^2b_+^2(1+q_k\cos m\pi) & 0 & 0 & a_+a_-b_+b_-(1+q_k\cos m\pi) \\ 0 & a_+^2b_-^2(1-q_k\cos m\pi) & a_+a_-b_+b_-(1-q_k\cos m\pi) & 0 \\ 0 & a_+a_-b_+b_-(1-q_k\cos m\pi) & a_-^2b_+^2(1-q_k\cos m\pi) & 0 \\ a_+a_-b_+b_-(1+q_k\cos m\pi) & 0 & 0 & a_-^2b_-^2(1+q_k\cos m\pi)\end{pmatrix}, \tag{111}$$

or equivalently, in the Fano-Bloch representation, as

$$\rho_{1(23\cdots k)} = \frac{1}{4}\sum_{\alpha\beta} R_{\alpha\beta}\sigma_\alpha\otimes\sigma_\beta \tag{112}$$

where the non vanishing matrix elements $R_{\alpha\beta}$ $(\alpha,\beta=0,1,2,3)$ are given by

$$R_{00}=1,\quad R_{11}=2\mathcal{N}^2\sqrt{(1-p^2)(1-p^{2(k-1)})},\quad R_{22}=-2\mathcal{N}^2\sqrt{(1-p^2)(1-p^{2(k-1)})}\;p^{n-k}\cos m\pi,$$

$$R_{33}=2\mathcal{N}^2(p^k+p^{n-k}\cos m\pi),\quad R_{03}=2\mathcal{N}^2(p^{k-1}+p^{n-k+1}\cos m\pi),\quad R_{30}=2\mathcal{N}^2(p+p^{n-1}\cos m\pi).$$

Following the standard procedure, presented here above for a two qubit system, the geometric discord is

$$D_{\rm g}(\rho_{1(23\cdots k)}) = \frac{1}{4}\,\min\{l_1+l_2, l_1+l_3, l_2+l_3\}. \tag{113}$$

where

$$l_1=R_{11}^2,\quad l_2=R_{22}^2,\quad l_3=R_{30}^2+R_{33}^2.$$

Explicitly, the quantities l_1, l_2 and l_3 are given by

$$l_1 = 4\mathcal{N}^4(1-p^2)(1-p^{2(k-1)}) \tag{114}$$

$$l_2 = 4\mathcal{N}^4(1-p^2)(1-p^{2(k-1)})p^{2(n-k)} \tag{115}$$

$$l_3 = 4\mathcal{N}^4\bigg[(1+p^{2(k-1)})(p^2+p^{2(n-k)}) + 4p^n\cos m\pi\bigg]. \tag{116}$$

Remarkably, the quantities (l_1, l_2, l_3) reduces (up to the overall multiplicative factor 2^{k-2}) to $\big((61),(62),(63)\big)$ for $k=2$, $\big((84),(85),(86)\big)$ for $k=3$ and $\big((106),(107),(108)\big)$ for $k=4$. Moreover, we have

$$D_{\rm g}(\rho_{1(23\cdots k)}) = \frac{1}{2^{k-2}}D_{\rm g}(\rho_{1|23\cdots k}). \tag{117}$$

It turns out that the two bi-partitioning schemes, discussed in this paper, characterize equivalently the pairwise geometric quantum discord in multi-partite Shrödinger cat states (6).

6 Concluding remarks

In this paper, we developed a general method to evaluate the pairwise geometric discord in a mixed multi-qubit states. We especially focused on reduced density matrices $\rho_{123\cdots k}$ obtained by a trace procedure from a balanced superpositions of n-qubit states possessing parity invariance and exchange symmetry. Closed analytical expressions for the geometric quantum discord based on Hilbert-Schmidt distance were derived. Two splitting scenarios were discussed. In the first one, a recursive algorithm is proposed to determine explicitly the pairwise geometric discord between the first qubit and the remaining $(k-1)$ qubits. For the Shrödinger cat sates considered in this work, the parity and exchange symmetries offer enormous advantages in determining the geometric measure of quantum discord. In particular, a recursive prescription, expressing the multi-component correlations tensors in terms of two-qubit correlation matrices elements, is obtained. This constitutes the key ingredient in deriving the pairwise geometric quantum discord. Another important issue we examined in this work concerns the explicit derivation of classical (zero discord) states. We also presented a second splitting scheme according to which a $(k-1)$-qubit cluster is viewed as single qubit so that the whole k-qubit system is mapped into a two-qubit system. We have shown that in both bi-partitioning schemes, the Hilbert-Schmidt distance characterize equivalently the bipartite quantum correlations. The second encoding scheme appears to be particularly convenient and more tractable from a computational point of view, contrarily to the first one which requires lengthy calculations. We believe that the results obtained in this work can be extended to other classes of multi-qubit states. Also, they can used in evaluating multipartite geometric quantum discord in the spirit of the results recently obtained in [45, 46]. Finally, another interesting issue, that deserves a special attention, concerns the distribution and the so-called monogamy property of geometric quantum discord between the different components in multi-partite Shrödinger cat states. In this sense, we believe that the results obtained in this work can be useful.

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