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Cover Letter

Dear Editor,

Please find attached the revised version of paper entitled

Unified approach of correlations using linear relative entropy

for publication in Physics Letters A.

Thank you very much.
Yours sincerely,

M. Daoud, R. Ahl Laamara and W.Kaydi.

Manuscript

A unified scheme of correlations using linear relative entropy

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Abstract

A linearized variant of relative entropy is used to quantify in an unified scheme the different kinds correlations in a bipartite quantum system. As illustration, we consider a two-qubit state with parity and exchange symmetry for which we derive the total, classical and quantum correlations. We also give the explicit forms of its closest product state, closest classical state and the corresponding closest product state to derive a closed additive relation involving the various correlations when measured by linear relative entropy.

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1 Introduction

Quantum entanglement in quantum systems, comprising two or more parts, constitutes a key concept to distinguish between quantum and classical correlations and subsequently to understand quantum-classical boundary. Also, besides its fundamental importance, entanglement is commonly accepted to be extremely important in the development of modern quantum information science [1, 2, 3, 4, 5, 6]. In fact, they have found various applications in quantum information processing as for instance quantum cryptography [7], quantum teleportation [8], quantum dense coding [9]. Nowadays, entanglement began to be recognized as valuable resource for performing communication and computational tasks [10, 11, 12]. In view of these remarkable realizations and implementations, the concept of entanglement is expected to have many other implications and applications in others areas of research, especially condensed matter physics.

Therefore, quantification and characterization of quantum correlations between the sub-components of a quantum system have attracted a special attention during the last two decades. Different measures were introduced from different perspectives and for various purposes [13, 14, 15, 16]. Probably the most familiar measure is quantum discord [17, 18] which goes beyond the entanglement of formation [19, 20]. It is given by the difference of total and classical correlations existing in a bipartite system. Now, it is well understood that almost all quantum states, including unentangled (separable) ones, posses quantum correlations. However, the analytical evaluation of quantum discord requires an optimization procedure that is generally a challenging task [21, 22, 23, 24, 25, 26, 27, 28]. To overcome this difficulty, a geometrical approach was proposed in [29]. This uses the Hilbert-Schmidt norm in the space of density matrices and presents the advantage that it provides closed analytical expressions. Clearly, Hilbert-Schmidt norm is not the unique distance which can be defined in the space of quantum states. Several distances are possible (trace distance, Bures distance, ...) with their own advantages and drawbacks and each one might be useful for some appropriate purpose [30, 31, 32, 33].

The states of any multipartite quantum system can be classified as being classical, quantum-classical and quantum states. Subsequently, the correlations can also be categorized in total, quantum, semi-classical and classical correlations. This requires a specific measure (entropic or geometric measure) to decide about the dissimilarity between a given quantum state and its closest one without the desired property and to provide a scheme to compare consistently different correlations existing in systems comprising two or more parts. In this sense, using the relative entropy, an approach unifying the correlations in multipartite systems was developed recently in [34]. In particular, a very significant and interesting additivity relation was reported ($D+C=T+L$) which reflects the sum of quantum D and classical C correlations is equal to the sum of total mutual correlations T and another quantity L that is exactly the difference between D and the quantum discord as originally introduced in [17, 18].

However, it must be noticed that despite its theoretical information meaning, the relative entropy is not symmetrical in its arguments and therefore can not be viewed as a true metric distance. In other hand, from an analytical point of view, the derivation of closed expressions of based entropy measures involves optimization procedures that are in general challenging and complicated to achieve. In this respect, a purely geometrical unified framework to classify the correlations in a given quantum state was discussed in [35, 36]. Using Hilbert-Schmidt norm and paralleling the analytical analysis in deriving the geometric discord, the geometric measures of mutual and classical correlations in a given system were derived [35, 36]. Contrarily to relative entropy measures, the additivity relation of the type $(D + C = T + L)$ is generally not satisfied when quantum correlations are quantified with Hilbert-Schmidt distance.

In this paper, we introduce a linearized variant of relative entropy to obtain analytical expressions of quantum and classical correlations in a two qubit system. The relation with the geometric measure based on Hilbert-Schmidt norm is established and allows the derivation of computable correlations. In this respect, this approach can be seen, in some sense, interpolating between the based relative entropy view and the geometrical one. More specifically, it provides us with a very simple way to perform the optimization required in deriving closest product, classical and classical product states. We also show that the correlations satisfy a closed additivity. Also, the analytical expressions of the correlations are explicitly derived.

This paper is organized as follows. In the second section, using the linear entropy, we define the linear form of relative entropy which can decomposed in symmetric and anti-symmetric parts. The antisymmetric part is related to quantum Jensen-Shannon divergence. In other hand, the symmetric part is exactly the Hilbert-Schmidt distance. We also discuss and compare the additivity relations of the various correlations in a bipartite quantum system using the linear relative entropy and Hilbert-Schmidt norm. In section 3, as illustration, we consider bipartite system possessing the parity symmetry and invariant under the exchange of its sub-components. In this situation the explicit derivation of closest product state, the classical state and its closest product state is achieved. The analytical expressions of total, quantum and classical correlations are obtained and the additivity relation are discussed. Concluding remarks close this paper.

2 Correlation quantifiers based symmetrized linear relative entropy

2.1 Correlation quantifiers based relative entropy

The main ingredient of the unified view of the correlations existing in multipartite systems is the concept of relative entropy [34]. It is the quantum analogue of the Kullback-Leibler divergence between two classical probability distributions and provides a measure of the dissimilarity between two quantum states. The relative entropy defined by

$$S(\rho\|\sigma) = -\mathrm{Tr}(\rho \log \sigma) - S(\rho), \tag{1}$$

constitutes a quantitative tool to distinguish between the states of a given quantum and gives the distance between them according to the nature of their properties ($S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy). The distance between a given state ρ and the closest product state $(\pi_\rho = \rho_A \otimes \rho_B)$, where ρ_A and ρ_B denote the reduced densities matrices of the subsystems, as measured by relative entropy, quantifies the total correlation $T = S(\rho\|\pi_\rho)$. It writes as the difference of the von Neumann [34]

$$T = S(\rho\|\pi_\rho) = S(\pi_\rho) - S(\rho) \tag{2}$$

Similarly, using relative entropy, the quantum discord encompassing quantum correlations is measured as the minimal distance between the state ρ and the classical states

$$\chi_\rho = \sum_{i,j} p_{i,j} |i\rangle\langle i| \otimes |j\rangle\langle j| \tag{3}$$

where $p_{i,j}$ are the probabilities and $\{|i\rangle, |j\rangle\}$ local basis. It writes also as the difference between the von Neumann entropies of the states ρ and χ_ρ [34]

$$D = S(\rho\|\chi_\rho) = S(\chi_\rho) - S(\rho). \tag{4}$$

The classical correlation, as measured by relative entropy, gives the distance between the classical state χ_ρ and its closest classical product state π_{χ_ρ}. It coincides with the difference of von Neumann enropies of the relevant states

$$C = S(\chi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\chi_\rho). \tag{5}$$

In this approach the based relative entropy quantum correlations or quantum discord D (4) does not coincide with the quantum discord as defined originally in [17, 18]. The difference is given by [34]

$$L = S(\pi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\pi_\rho). \tag{6}$$

Noticing that the based entropy correlations T, D, C and L can be expressed as differences of von Neumann entropies (Eqs. (2), (4), (5) and (6)), Modi et al have shown the following remarkable additivity relation [34]

$$T - D - C + L = 0. \tag{7}$$

It must be noticed that the relative entropy (1) is not symmetric under the exchange $\rho \leftrightarrow \sigma$. In this respect, it cannot define a distance from a purely mathematical point of view. Moreover, the relative entropy induces intractable minimization procedures that are in general very difficult to perform. To avoid such difficulties the linear relative entropy offers an alternative way to get computable expressions of correlations existing in multipartite systems [35].

2.2 Symmetrized linear relative entropy

The linear entropy

$$S_2(\rho) \doteq 1 - \mathrm{Tr}(\rho^2)$$

is related to the degree of purity, $P = \mathrm{Tr}(\rho^2)$, and therefore reflects the mixedness in the state ρ. It is defined as a linearized variant of von Neumann entropy by approximating $\log\rho$ by $\rho - \mathbb{I}$ where $\mathbb{I}$ stands for the identity matrix. Accordingly, the relative entropy (1) can be linearized as follows

$$S_l(\rho_1\|\rho_2) = \mathrm{Tr}\rho_2(\rho_1 - \rho_2) \tag{8}$$

that is obviously not symmetric by interchanging ρ_1 and ρ_2. To define a symmetrized linear relative entropy, $S_l(\rho_1\|\rho_2)$ is decomposed as the sum of two terms: symmetric and antisymmetric. The symmetric part is defined by

$$S_+(\rho_1\|\rho_2) = S_l(\rho_1\|\rho_2) + S_l(\rho_2\|\rho_1). \tag{9}$$

The antisymmetric term is given by

$$S_-(\rho_1\|\rho_2) = S_l(\rho_1\|\rho_2) - S_l(\rho_2\|\rho_1) \tag{10}$$

and rewrites as the differences between the linear entropies of the states ρ_1 and ρ_2

$$S_-(\rho_1\|\rho_2) = S_2(\rho_2) - S_2(\rho_1). \tag{11}$$

It is important to emphasize that the symmetrized linear relative entropy (9) is related to some generalized version of relative entropy discussed in the literature. Indeed, it can be expressed as

$$S_-(\rho_1\|\rho_2) = D_2(\rho_1 + \rho_2\|\rho_2 - \rho_1) - D_2(\rho_1 + \rho_2\|\rho_1 - \rho_2) \tag{12}$$

in terms of quantum Jensen-Shannon entropy of order 2 defined by

$$D_2(\rho_1\|\rho_2) := S_2\left(\frac{\rho_1 + \rho_2}{2}\right) - \frac{1}{2}S_2(\rho_1) - \frac{1}{2}S_2(\rho_2). \tag{13}$$

which is a symmetrized version of relative entropy. It was recently used to investigate the distance between quantum states (see for instance [37, 38] and references quoted therein) and subsequently constitutes a good geometric candidate to classify quantum states according to their correlation contents. The square root of quantum Jensen-Shannon divergence is a metric and can be isometrically embedded in a real Hilbert space equipped with a Hilbert-Schmidt norm [37]. This result is very useful in our context. In fact, using (8) and (9), the symmetric part of symmetrized linear relative entropy is exactly the Hilbert-Schmidt distance

$$S_+(\rho_1\|\rho_2) = \|\rho_1 - \rho_2\|^2. \tag{14}$$

The symmetric and antisymmetric linear entropy are the essential ingredients in this work. The symmetrized linear relative entropy is utilized to measure the distance between the states of a given quantum system and the antisymmetrical linear relative entropy quantifies the amount of correlations existing between two distinct states. Hence, the linear relative entropy offers an adequate scheme to derive explicit expressions for correlations in a common framework and to discuss the relationship between quantum, semi-quantum and classical correlation. Further, in view of the emergence of Hilbert-Schmidt distance in the context of linear relative entropy, interesting relations between the correlations as measured by linear relative entropy and their Hilbert-Schmidt counterparts, as defined in [35, 36], can be derived. This issue is discussed in the remaining part of this section.

2.3 Additivity relation of geometric and entropic correlations

The Fano-Bloch representation of an arbitrary two-qubit state ρ is

$$\rho = \frac{1}{4}\sum_{\alpha,\beta} R_{\alpha,\beta}\, \sigma_\alpha \otimes \sigma_\beta \tag{15}$$

where $\alpha, \beta = 0, 1, 2, 3,\ R_{i0} = {\rm Tr}\rho(\sigma_i \otimes \sigma_0),\ R_{0i} = {\rm Tr}\rho(\sigma_0 \otimes \sigma_i)$ are components of local Bloch vectors and $R_{ij} = {\rm Tr}\rho(\sigma_i \otimes \sigma_j)$ are components of the correlation tensor. The operators σ_i $(i = 1, 2, 3)$ stand for the three Pauli matrices and σ_0 is the identity matrix. The distance (14), between two distinct density matrices ρ and ρ', writes as

$$S_+(\rho\|\rho') \equiv d(\rho, \rho') = \frac{1}{4}\sum_{\alpha,\beta} (R_{\alpha,\beta} - R'_{\alpha,\beta})^2, \tag{16}$$

in terms of the elements of the correlations matrices. Beside the distance defined by (16), the linear analogue of total correlation T (2), quantum correlation D (4), Classical correlation C (5) and the quantity L (6) are respectively given by

$$T_2 = S_-(\rho\|\pi_\rho) \qquad D_2 = S_-(\rho\|\chi_\rho) \qquad C_2 = S_-(\chi_\rho\|\pi_{\chi_\rho}) \qquad L_2 = S_-(\pi_\rho\|\pi_{\chi_\rho}). \tag{17}$$

Using the expression (11), it is simply verified that the correlations T_2, D_2, D_2 and L_2 can be written as differences of linear entropies. This implies the remarkable additivity relation

$$T_2 - D_2 - C_2 + L_2 = 0. \tag{18}$$

Since the Hilbert-Schmidt provides an useful tool to quantify geometrically the quantum correlation (geometric quantum discord) [29], the geometric analogues of total correlation T (2), quantum correlation D (4), classical correlation C (5) and the quantity L (6) were introduced in [36]. They are defined by

$$T_{\rm g} \equiv \|\rho - \pi_\rho\|^2, \qquad C_{\rm g} \equiv \|\chi_\rho - \pi_{\chi_\rho}\|^2, \qquad D_{\rm g}(\rho) = \|\rho - \chi_\rho\|^2, \qquad L_{\rm g} \equiv \|\pi_\rho - \pi_{\chi_\rho}\|^2, \tag{19}$$

Furthermore, in view of the relation between the distance (9) and the Hilbert-Schmidt norm given by (14), the based linear relative entropy correlations can be expressed in terms of their geometric counterpart. Indeed, from the definitions (17), one gets

$$T_2 = T_{\rm g} - 2S_2(\pi_\rho\|\rho), \quad D_2 = D_{\rm g} - 2S_2(\chi_\rho\|\rho), \quad C_2 = C_{\rm g} - 2S_2(\pi_{\chi_\rho}\|\chi_\rho), \quad L_2 = L_{\rm g} - 2S_2(\pi_{\chi_\rho}\|\pi_\rho) \tag{20}$$

or alternatively

$$T_2 = T_{\rm g} + 2{\rm Tr}\big(\pi_\rho(\rho - \pi_\rho)\big),\ D_2 = D_{\rm g} + 2{\rm Tr}\big(\chi_\rho(\rho - \chi_\rho)\big),\ C_2 = C_{\rm g} + 2{\rm Tr}\big(\pi_{\chi_\rho}(\chi_\rho - \pi_{\chi_\rho})\big),\ L_2 = L_{\rm g} - 2{\rm Tr}\big(\pi_{\chi_\rho}(\pi_\rho - \pi_{\chi_\rho})\big) \tag{21}$$

Reporting the equations (20), or equivalently (21), in the additivity relation (18), it easily verified that the correlations as measured by Hilbert-Schmidt norm satisfy the relation

$$T_{\rm g} - D_{\rm g} - C_{\rm g} + L_{\rm g} = \Delta_{\rm g} \tag{22}$$

where

$$\Delta_{\rm g} = 2\bigg[{\rm Tr}\big(\pi_\rho(\pi_\rho - \rho)\big) + {\rm Tr}\big(\pi_{\chi_\rho}(\chi_\rho - \pi_\rho)\big)\bigg]. \tag{23}$$

It is important to note the classical states χ_ρ satisfy ${\rm Tr}\rho\chi_\rho = {\rm Tr}\chi_\rho^2$ [29] and subsequently the geometric discord $D_{\rm g}$ coincides with the linear quantum correlation D_2 ($D_2 = D_{\rm g}$). It is clear from equation (22) that the geometric measures of correlations close an additive relation of type (18) when $\Delta_{\rm g}$ vanishes. The use of Hilbert-Schmidt norm in quantifying and unifying the different correlations in a bipartite system was recently investigated in two-qubit X states [36]. More specifically, it has been shown that the additivity relation does not hold in general except for some specific cases like for instance Bell states [35]. Along the same lines of reasoning, we shall consider a family of two-qubit states parameterized by two real parameters to derive the explicit from of pairwise correlations measured by linear relative entropy and we compare the obtained results with ones measured by Hilbert-Schmidt norm. This constitutes the main of the next section.

3 Analytical expressions of correlations

To illustrate the results discussed in the previous section and to investigate qualitatively the differences between the linear relative entropy measures and geometric ones based on Hilbert-Schmidt distance, we shall consider a family of two qubit density matrices whose entries are specified in terms of two real parameters. They are defined as

$$\rho = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1c_2} \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ \sqrt{c_1c_2} & 0 & 0 & c_2 \end{pmatrix} \tag{24}$$

in the computational basis $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The parameters c_1 and c_2 are positive with $c_1 + c_2 \leq 1$. We have taken all entries positives. In fact, the local unitary transformation

$$|0\rangle_k \to \exp\left(\frac{i}{2}(\theta_1 + (-)^k\theta_2)\right)|0\rangle_k$$

eliminates the phase factors of the off diagonal elements and the rank of the density matrix ρ remains unchanged. Re-expressed in the Fano-Bloch representation, the density ρ takes the form (15) and the non vanishing matrix correlation elements are

$$R_{30} = R_{03} = c_1 - c_2 \qquad R_{33} = 2(c_1 + c_2) - 1, \tag{25}$$

$$R_{11} = 1 - (\sqrt{c_1} - \sqrt{c_2})^2 \qquad R_{22} = 1 - (\sqrt{c_1} + \sqrt{c_2})^2 \tag{26}$$

where $0 \leq c_1, c_2 \leq 1$. On other hand, the density matrix considered here is invariant under parity symmetry and exchange transformation (ρ commutes with $\sigma_3 \otimes \sigma_3$ and the permutation operator which exchanges the qubit state $|i,j\rangle$ to $|j,i\rangle$). This reduces considerably the analytical evaluations of bipartite correlations.

3.1 Total correlation and closest product state

3.2 Closest product state

We shall begin our illustration by deriving the explicit expression of total correlation T_2 defined by (17). So, we first determine the closest product state to the density matrix ρ (24). An arbitrary product state $\pi_\rho = \rho_1 \otimes \rho_2$ writes

$$\pi_\rho = \rho_1 \otimes \rho_2 = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3}(a_i\sigma_i \otimes \sigma_0 + b_i\sigma_0 \otimes \sigma_i) + \sum_{i,j=1}^{3} a_ib_j\sigma_i \otimes \sigma_j\right] \tag{27}$$

where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ denote the unit Bloch vectors of the states ρ_1 and ρ_2

$$\rho_1 \;=\; \frac{1}{2}[\sigma_0 + \sum_{i=1}^{3} a_i\sigma_i], \qquad \rho_2 \;=\; \frac{1}{2}[\sigma_0 + \sum_{i=1}^{3} b_i\sigma_i]. \tag{28}$$

Since the density matrix ρ is invariant by exchanging the role of the subsystems 1 and 2, its closest product state is also invariant under this operation. This implies

$$a_i = b_i \qquad i = 1, 2, 3.$$

Furthermore, the parity symmetry of the density matrix ρ ($[\rho, \sigma_3 \otimes \sigma_3] = 0$) implies the parity invariance of the product π_ρ. This imposes

$$a_i = b_i = 0 \qquad i = 1, 2.$$

It follows that the distance between the state ρ and π_ρ takes the simple form

$$d(\rho, \pi_\rho) \;=\; \frac{1}{4}\left[2(R_{30} - a_3)^2 + R_{11}^2 + R_{22}^2 + (R_{33} - a_3^2)^2\right] \tag{29}$$

to be optimized with respect one variable only, i.e. a_3. The parity and exchange symmetries simplify the minimization process to get the closest product state. Indeed, it is easy to see that the minimum value of the distance (29) is reached when the variable a_3 satisfies the following cubic equation

$$a_3^3 + a_3(1 - R_{33}) - R_{30} = 0. \tag{30}$$

Being constrained to real solutions, the only real solution is given by

$$a_3 = \sqrt[3]{\frac{\sqrt{\Delta} + R_{30}}{2}} - \sqrt[3]{\frac{\sqrt{\Delta} - R_{30}}{2}} \tag{31}$$

where

$$\Delta = R_{30}^2 + \frac{4}{27}(1 - R_{33})^3$$

is positive ($R_{33} \leq 1$). It follows that the closest product state to ρ is explicitly given by

$$\pi_\rho \;=\; \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + a_3\ \sigma_3 \otimes \sigma_0 + a_3\ \sigma_0 \otimes \sigma_3 + a_3^2\ \sigma_3 \otimes \sigma_3\right]. \tag{32}$$

3.2.1 Total correlation

Using (17) together with (10), the total correlation, as measured by linear relative entropy, writes

$$T_2 = \frac{1}{4}[2(R_{03}^2 - a_3^2) + R_{11}^2 + R_{22}^2 + (R_{33}^2 - a_3^4)]. \tag{33}$$

The behavior of total correlation T_2 versus c_1 is given in the figure 1 for different values of $\alpha = c_1 + c_2$ ($\alpha = 0.1, 0.2 \cdots, 0.9$). In this figure, as well as in others presented in this paper, the parameter c_1 is varying from 0 to α. Accordingly, it is easily distinguishable the line representing the behavior of total correlation T_2 versus c_1 for each fixed value of α. For instance, the short line corresponds to $\alpha = 0.1$ and the long one represents the total correlation when $\alpha = 0.9$. The minimal values of total correlation are obtained for ($c_1 = 0, c_2 = \alpha$) and ($c_1 = \alpha, c_2 = 0$). These two situations correspond respectively to states of the form

$$\rho(c_1 = 0, c_2 = \alpha) = \alpha|11\rangle\langle 11| + (1-\alpha)|\psi_1\rangle\langle\psi_1| \tag{34}$$

and

$$\rho(c_1 = \alpha, c_2 = 0) = \alpha|00\rangle\langle 00| + (1-\alpha)|\psi_1\rangle\langle\psi_1| \tag{35}$$

where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{36}$$

The total correlation reaches its maximal values for $c_1 = c_2 = \frac{\alpha}{2}$. In this respect, among the states under consideration, more correlated are those of the form

$$\rho(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}) = \alpha\rho_0 + (1-\alpha)\rho_1 \tag{37}$$

where the states ρ_1 and ρ_0 are respectively given by

$$\rho_1 = |\psi_1\rangle\langle\psi_1| \tag{38}$$

where $|\psi_1\rangle$ is given by (36) and

$$\rho_0 = |\psi_0\rangle\langle\psi_0| \tag{39}$$

where $|\psi_0\rangle$ is the state defined by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

In other hand, it is clear from figure 1 that when $0 \leq \alpha \leq 0.5$, the total correlation T_2 increases as the parameter α increases. For instance, for $c_1 = 0.05$ the amount of classical correlation, present in states (24) with $\alpha = 0.1$, exceeds ones measured by linear entropy in states with $\alpha = 0.2, 0.3, 0.4, 0.5$. The situation is completely different for $\alpha \geq 0.5$. Indeed, for small values of c_1, the total correlation present in states with $\alpha = 0.6$ is higher than correlations present in states with $\alpha = 0.7, 0.8, 0.9$. For high values of c_1 ($c_1 = 0.55$ for instance), more correlation is obtained for states with $\alpha = 0.9$.

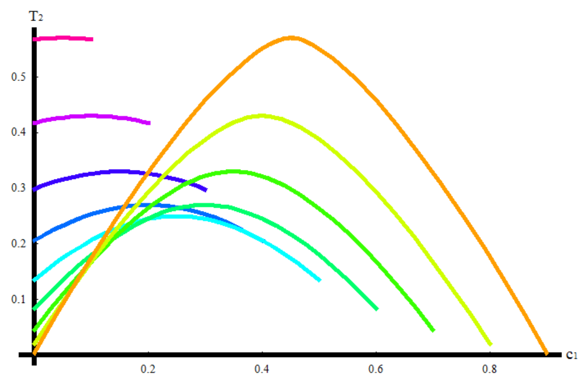

Figure 1. Total correlation T_2 versus the parameter c_1 for different values of $\alpha = c_1 + c_2$.

3.3 Quantum correlation and closest classical state

3.3.1 Quantum discord

It is commonly accepted that the explicit evaluation of based entropy quantum discord [17, 18] is a difficult task for an arbitrary bipartite system. This difficulty originates from the minimization procedure of conditional entropy which was achieved only for some special types of two qubit systems [21, 22, 23, 24, 39, 40, 41] (see also [42, 43] and references therein). To overcome this problem, the geometric quantifier of quantum discord, using the Hilbert-Schmidt norm, was introduced in [29]. It is defined as the minimal Hilbert-Schmidt distance between a given state ρ and the closest classical states of the form

$$\chi = \sum_{i=1,2} p_i |\psi_i\rangle\langle\psi_i| \otimes \rho_i \tag{40}$$

when the measurement is performed on the first subsystem. In equation (40), p_i is a probability distribution, ρ_i is the marginal density matrix of the second subsystem and $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an arbitrary orthonormal vector set. Based on the results obtained in [29], the explicit expression of the geometric quantum discord in the state (24) writes

$$D_{\rm g} = \frac{1}{4}\left(R_{11}^2 + R_{22}^2 + R_{33}^2 + R_{03}^2 - \lambda_{\rm max}\right) \tag{41}$$

where the correlation elements are given by (25) and (26) and $\lambda_{\rm max}$ is

$$\lambda_{\rm max} = \max(\lambda_1, \lambda_2, \lambda_3) \tag{42}$$

where λ_1, λ_2 and λ_3 denote respectively the elements of the diagonal matrix K given by

$$K := {\rm diag}\ (R_{11}^2, R_{22}^2, R_{33}^2 + R_{03}^2).$$

The closest classical state is the eigenstate associated with $\lambda_{\rm max}$ [29]. As already mentioned the quantum correlation D_2 coincides with the geometric measure of quantum discord. Thus, one gets

$$D_2 = D_{\rm g} = \frac{1}{4}\ \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \tag{43}$$

where the explicit expressions of the eigenvalues of the matrix K, corresponding to the state ρ, are

$$\lambda_1 = \left[1 - (\sqrt{c_1} - \sqrt{c_2})^2\right]^2$$

$$\lambda_2 = \left[1 - (\sqrt{c_1} + \sqrt{c_2})^2\right]^2$$

$$\lambda_3 = \frac{1}{2}\left[(3c_1 + c_2 - 1)^2 + (c_1 + 3c_2 - 1)^2\right]$$

Since λ_1 is always greater than λ_2, the geometric discord is simply given by

$$D_{\rm g} = \frac{1}{4}\min\{\lambda_1 + \lambda_2\ ,\ \lambda_2 + \lambda_3\} = \frac{1}{4}\ \{\min(\lambda_1, \lambda_3) + \lambda_2\}.$$

Thus, to compare the eigenvalues λ_1 and λ_3, we need to determine the sign of the following two variables function

$$\lambda_3 - \lambda_1 = 2(\sqrt{c_1} + \sqrt{c_2})(\sqrt{c_1}(2c_1 - 1) + \sqrt{c_2}(2c_2 - 1)),$$

which is positive when the parameters c_1 and c_2 satisfy the condition

$$\sqrt{c_1}(2c_1 - 1) + \sqrt{c_2}(2c_2 - 1) \geq 0. \tag{44}$$

Conversely, when this quantity is non positive, we have $\lambda_3 \leq \lambda_1$. Setting

$$\sqrt{c_1} = e^{-r}\cos\theta, \quad \sqrt{c_2} = e^{-r}\sin\theta \quad \text{with} \quad r \in \mathbb{R}, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

the condition (44) rewrites

$$e^{-r}(\cos\theta + \sin\theta)(2e^{-2r}(1 - \cos\theta\sin\theta) - 1) \geq 0$$

which is satisfied when

$$2e^{-2r}(1 - \cos\theta\sin\theta) - 1 \geq 0$$

or equivalently

$$c_1 + c_2 - \sqrt{c_1 c_2} \geq \frac{1}{2} \tag{45}$$

in terms of the parameters c_1 and c_2. The set of states of type (24) can be written as

$$\{\rho \equiv \rho_{c_1,c_2} \quad 0 \leq c_1 + c_2 \leq 1\} = \bigoplus_{\alpha=0}^{1}\{\rho_\alpha \equiv \rho_{c_1,\alpha-c_1} \quad 0 \leq c_1 \leq \alpha\}$$

with

$$c_1 + c_2 = \alpha \qquad 0 \leq \alpha \leq 1.$$

The condition (45) is satisfied if and only if $\alpha \geq \frac{1}{2}$. This implies that for a fixed value $\alpha \leq \frac{1}{2}$, the difference $\lambda_3 - \lambda_1$ is non positive and the geometric measure of quantum discord (43) writes

$$D_{\rm g} = D_{\rm g}^{+} = \frac{1}{4}\ (\lambda_2 + \lambda_3). \tag{46}$$

For $\alpha \geq \frac{1}{2}$, the condition (45) is satisfied for

$$0 \leq c_1 \leq \alpha_- \quad \alpha_+ \leq c_1 \leq \alpha$$

where

$$\alpha_{\pm} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{(1-\alpha)(3\alpha-1)} \tag{47}$$

In this situation, the geometric quantum discord is given by

$$D_{\rm g} = D_{\rm g}^{-} = \frac{1}{4}\,(\lambda_1 + \lambda_2). \tag{48}$$

It reads

$$D_{\rm g} = D_{\rm g}^{+} = \frac{1}{4}\,(\lambda_2 + \lambda_3). \tag{49}$$

when $\alpha_- \leq c_1 \leq \alpha_+$.

3.3.2 Closest classical state

To obtain the explicit form of the closest classical state to the state (24), we follow the procedure developed in [29]. It consists in minimizing the Hilbert-Schmidt distance and determining the eigenvector corresponding to the eigenvalue $\lambda_{\rm max}$. We discuss separately the situations $\lambda_{\rm max} = \lambda_1$ and $\lambda_{\rm max} = \lambda_3$. We first consider the situation where $\lambda_1 \leq \lambda_3$. In this case, we find that the zero discord states, as measured by Hilbert-Schmidt distance, has the form

$$\chi_\rho^- = \frac{1}{4}[\sigma_0 \otimes \sigma_0 + R_{30}\sigma_3 \otimes \sigma_0 + R_{30}\sigma_0 \otimes \sigma_3 + R_{33}\sigma_3 \otimes \sigma_3], \tag{50}$$

where the notation $-$ stands for the condition $\lambda_1 - \lambda_3 \leq 0$. In this case the pairwise quantum correlation is

$$D_2^- = S_-(\rho\|\chi_\rho^-) = \frac{1}{4}(\lambda_1 + \lambda_2) = \frac{1}{4}(R_{11}^2 + R_{22}^2), \tag{51}$$

which rewrites in terms of the parameters c_1 and c_2 as

$$D_2^- \equiv D_{\rm g}^-(\rho) = \frac{1}{4}\big[1 - (\sqrt{c_1} - \sqrt{c_2})^2\big]^2 + \frac{1}{4}\big[1 - (\sqrt{c_1} + \sqrt{c_2})^2\big]^2$$

It is interesting to note that the closest classical state χ_ρ^- satisfies

$$\mathrm{Tr}\rho\chi_\rho^- = \mathrm{Tr}\chi_\rho^{-2}$$

reflecting that the geometric quantum discord coincides indeed with the quantum correlation evaluated by means of linear relative entropy. Similarly, in the situation where $\lambda_1 > \lambda_3$, it is easy to verify that the closest classical state is given by

$$\chi_\rho^+ \quad = \quad \frac{1}{4}\Big[\sigma_0 \otimes \sigma_0 + R_{03}\sigma_0 \otimes \sigma_3 + R_{11}\sigma_1 \otimes \sigma_1\Big], \tag{52}$$

where the notation $^+$ refers now to the situation where $\lambda_1 - \lambda_3 > 0$. In this case, the maximal eigenvalue of the correlation matrix K is λ_1 and the Hilbert-Schmidt distance between the density matrix ρ and its closest classical state is

$$D_2^+ = S_-(\rho\|\chi_\rho^+) = \frac{1}{4}(\lambda_2 + \lambda_3) = \frac{1}{4}(R_{22}^2 + R_{03}^2 + R_{33}^2) \tag{53}$$

which can be also re-expressed as

$$D_2^+ \equiv D_{\rm g}^+(\rho) = \frac{1}{4}\bigg[\big[1-(\sqrt{c_1}+\sqrt{c_2})^2\big]^2 + \frac{1}{2}\big[(3c_1+c_2-1)^2+(c_1+3c_2-1)^2\big]\bigg].$$

Here also, the closest classical states (50) and (52) satisfy the following identities

$$\mathrm{Tr}\rho\chi_\rho^\pm = \mathrm{Tr}\chi_\rho^{\pm^2}$$

The quantum discord as measured by linear entropy, which coincides with geometric quantum discord using Hilbert-Schmidt norm, is represented in the figures 2 and 3. Figure 2 gives the amount of quantum correlations for states with $\alpha \leq \frac{1}{2}$. It is clearly seen that the quantum discord is minimal value for $c_1 = \frac{\alpha}{2}$. It is intersting to note that in the minimally discordant states, given by (37), the total correlation is maximal (see figure 1). In other hand, the maximal value of quantum discord is obtained for states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ which are respectively given by the expressions (34) and (35). Here also it important to note that the total correlation, in these maximally discord states, is minimal (see figure 1). It follows that for the states under consideration(24) the quantum discord is maximal (resp. minimal) for states presenting a minimal (resp. maximal) amount of total correlation. The quantum discord evolves smoothly contrarily to the situation where $\alpha \geq \frac{1}{2}$ (figure 3) where the quantum discord changes suddenly when $c_1 = \alpha_-$ and $c_1 = \alpha_+$ given by the expressions (47). This sudden change of quantum discord occurs when the states present a maximum amount of quantum correlation. The behavior of quantum discord present then three distinct phases: $0 \leq c_1 \leq \alpha_-$, $\alpha_- \leq c_1 \leq \alpha_+$ and $\alpha_- \leq c_1 \leq \alpha$. The minimal value of quantum discord is obtained in the intermediate phase ($\alpha_- \leq c_1 \leq \alpha_+$) for the states of the form (37).

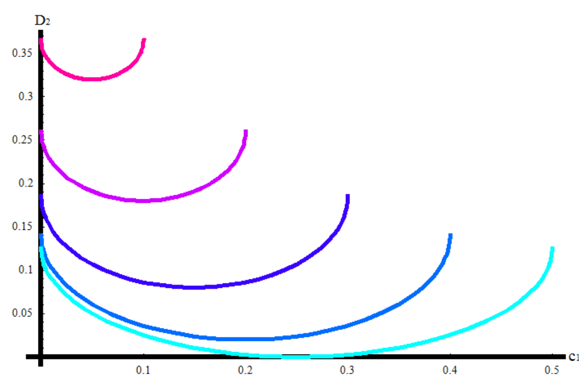

Figure 2. Quantum discord $D_2 \equiv D_{\rm g}$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$.

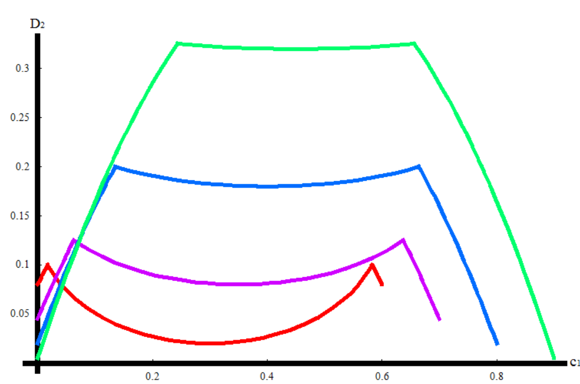

Figure 3. Quantum discord $D_2 \equiv D_{\rm g}$ as function of the parameter c_1 for $\alpha \geq \frac{1}{2}$.

3.4 Classical correlations

Now, we consider the analytical derivation of classical correlation (17) in the state ρ (24). For this end, we need to determine the closest product states to classical states χ_ρ^- and χ_ρ^+. We discuss first the situation where the classical is given by χ_ρ^- (50). Noticing that χ_ρ^- possess parity and exchange symmetries, its closest product state is obtained following the method used to derive the closest product state (32). Hence, one gets

$$\pi_{\chi_\rho^-} = \frac{1}{4}\bigg[\sigma_0 \otimes \sigma_0 + a_3\sigma_3 \otimes \sigma_0 + a_3\sigma_0 \otimes \sigma_3 + a_3^2\sigma_3 \otimes \sigma_3\bigg] \tag{54}$$

that coincides with π_ρ. Subsequently, the classical correlation writes

$$C_2^- = \frac{1}{4}[2(R_{03}^2 - a_3^2) + (R_{33}^2 - a_3^4)]. \tag{55}$$

The determination of the closest classical product to classical state χ_ρ^+ is slightly different from the previous case. In fact, the state χ_ρ^+ is invariant under parity transformation but it is not invariant under exchange symmetry. It follows that the closest classical product should have the form

$$\pi_{\chi_\rho}^+ \quad = \quad \frac{1}{4}\bigg[\sigma_0 \otimes \sigma_0 + \alpha_3\sigma_3 \otimes \sigma_0 + \beta_3\sigma_0 \otimes \sigma_3 + \alpha_3\beta_3\sigma_3 \otimes \sigma_3\bigg], \tag{56}$$

where the variables α_3 and β_3 are obtainable by minimizing the Hilbert-Schmidt distance between the states χ_ρ^+ (52) and $\pi_{\chi_\rho^+}$ (56). This gives

$$\alpha_3 = 0 \qquad \beta_3 = R_{03}.$$

Thus, the closest product state is

$$\pi_{\chi_\rho^+} = \frac{1}{4}[\sigma_0 \otimes \sigma_0 + R_{03}\sigma_0 \otimes \sigma_3], \tag{57}$$

and from the definition (17), the classical correlation reads

$$C_2^+ = \frac{1}{4}R_{11}^2. \tag{58}$$

Finally, using the definition of the quantity L_2 (17) and the expressions of the closest product π_ρ (32) and the closest classical product states $\pi_{\chi_\rho^-}$ (54) and $\pi_{\chi_\rho^+}$ (57), one obtains

$$L_2^- = 0 \qquad L_2^+ = \frac{1}{4}[2a_3^2 + a_3^4 - R_{03}^2], \tag{59}$$

and one recovers the additivity relation

$$T_2 + L_2^{\pm} = D_2^{\pm} + C_2^{\pm}$$

as excepted.

Figures 4 and 5 give the classical correlations the distances between classical states and their closest product states as measured by linear relative entropy. For $\alpha \leq \frac{1}{2}$, the classical correlation behaves like total correlation. It is maximal for the states satisfying $c_1 = c_2 = \frac{\alpha}{2}$ (37) and minimal for $(c_1 = 0, c_2 = \alpha)$ (Eq. (34)) and $(c_1 = \alpha, c_2 = 0)$ (Eq. (35)). Figure 5 shows a discontinuity of classical correlations in the points $c_1 = \alpha_-$ and $c_1 = \alpha_+$ (Eq.(47)) where the quantum discord changes suddenly. This discontinuity indicates that the linear relative entropy L_2^+, between the product states π_ρ and the product of classical states π_{χ_ρ}, is non-vanishing and the total correlation does coincides with the sum of quantum discord D_2^+ and classical correlation C_2^+ when the parameter c_1 ranges from α_- to α_+.

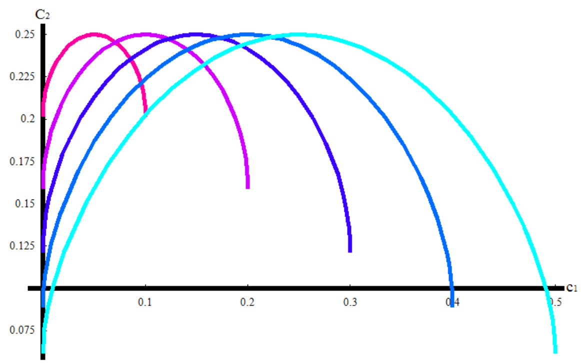

Figure 4. Classical correlations C_2 versus c_1 for $\alpha \leq \frac{1}{2}$.

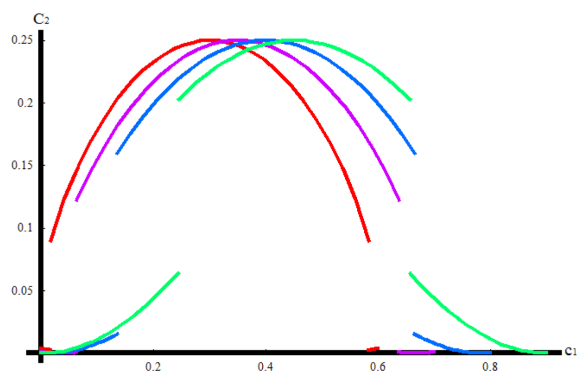

Figure 5. Classical correlations C_2 versus c_1 for $\alpha \geq \frac{1}{2}$.

3.5 Hilbert-Schmidt measures of correlations

The equations (20), or equivalently (21), give the relations between the geometric Hilbert-Schmidt measures of the various correlations present in the states under consideration and based linear relative entropy correlations. Indeed, using the expressions of closest product states π_ρ (32), one gets

$$\mathrm{Tr}\big(\pi_\rho(\pi_\rho - \rho)\big) = \frac{1}{4}a_3^2(R_{33} - a_3^2). \tag{60}$$

Similarly, using the closest classical state χ_ρ^- (50), χ_ρ^+ (52) and the closest classical product states $\pi_{\chi_\rho^-}$ (54), $\pi_{\chi_\rho^+}$ (57), one shows

$$\mathrm{Tr}\big(\pi_{\chi_\rho^-}(\chi_\rho^- - \pi_\rho)\big) = \frac{1}{4}a_3^2(a_3^2 - R_{33}) \tag{61}$$

and

$$\mathrm{Tr}\big(\pi_{\chi_\rho^+}(\chi_\rho^+ - \pi_\rho)\big) = \frac{1}{4}R_{03}(R_{03} - a_3) \tag{62}$$

for $\lambda_1 \leq \lambda_3$ and $\lambda_1 > \lambda_3$ respectively. Reporting (60) in (21) and using the result (33), one obtains the following expression of the geometric measure of total correlation

$$T_g = \frac{1}{4}\Big[2(R_{30} - a_3)^2 + R_{11}^2 + R_{22}^2 + (R_{33} - a_3^2)^2\Big]. \tag{63}$$

This result can be also derived from (19) using the expressions of the closest product state (32). In the same way, substituting (61) (resp. (62)) in the relevant expression in (21) and using (55) (resp. (58)), one gets

$$C_{\rm g}^- = \frac{1}{4}\big(2(R_{30} - a_3)^2 + (R_{33} - a_3^2)^2\big). \tag{64}$$

and

$$C_{\rm g}^+ = \frac{1}{4}\lambda_1 = \frac{1}{4}R_{11}^2. \tag{65}$$

The expressions of the quantities $L_2^\pm$ (59) together with the equation (21), defining the relation between $L_2^\pm$ and $L_{\rm g}^\pm$, yield

$$L_{\rm g}^- = 0 \qquad L_{\rm g}^+ = \frac{a_3^2}{4}\big(1 + a_3^2 + a_3(a_3^2 - R_{33})^2\big) \tag{66}$$

Finally, using the equations (60), (61) and (62), the quantity Δ_g, defined by (23), is given by

$$\Delta_{\rm g}^- = 0 \qquad \Delta_{\rm g}^+ = \frac{1}{2}a_3^2(R_{33} - a_3^2) \tag{67}$$

for $\lambda_1 \leq \lambda_3$ and $\lambda_1 > \lambda_3$ respectively. From the results (51), (63), (64), (66) and (67), one verifies that

$$T_{\rm g}^- - D_{\rm g}^- - C_{\rm g}^- = 0$$

where $T_{\rm g}^-$ stands for the total correlation $T_{\rm g}$ for the two-qubit states ρ labeled by the parameters c_1 and c_2 fulfilling the condition $\lambda_1 \leq \lambda_3$. This reflects that the total geometric correlation present in this sub-class of states is exactly the sum of geometric quantum and classical correlations. This result is no longer valid for the states such that $\lambda_3 < \lambda_1$. Indeed, from the equations (53), (63), (65), (66) and (67), we have

$$T_{\rm g}^+ - D_{\rm g}^+ - C_{\rm g}^+ + L_{\rm g}^+ = \Delta_{\rm g}^+.$$

Using the equations (66) and (67), one verifies

$$\Delta_{\rm g}^+ - L_{\rm g}^+ = -\frac{1}{4}a_3^2\Big(a_3^2 + (a_3^2 + 1 - R_{33})^2\Big) \tag{68}$$

which implies that

$$T_{\rm g}^+ - D_{\rm g}^+ - C_{\rm g}^+ \leq 0.$$

In the last equation, the equality holds for $a_3 = 0$ which gives $R_{30} = 0$ and subsequently the state ρ (24) becomes a two-qubit state of Bell type. This agrees with the result derived in [36].

4 Concluding remarks

The relative entropy constitutes a typical measure providing a quantitative ingredient to deal, in a common framework, with the different kind of correlations in multipartite systems [34]. In this view a closed additivity relation involving total T, quantum D, classical C correlations and L the relative entropy between a classical state and its closest classical product states. However, the relative entropy formalism presents some technical inconvenience when one needs to determine analytic expressions of correlations. This is mainely due to optimization process required in minimizing the distance between a quantum state and its closest one without the required property. To overcome this problem, and paralleling the definition of geometric discord, the Hilbert-Schmidt distance was considered to introduce the geometric variants of total, quantum, classical correlations [35, 36]. Unfortunately, the closed additive relation, obtained when the correlations are measured by relative entropy, ceases in general to be valid when the Hilbert-Schmidt norm is used. In this paper, we proposed an unified scheme based on a linearized variant of relative entropy to quantify the correlation in a bipartite quantum system. This provided us with an useful tool to get computable expressions and to classify the different correlations in a bipartite quantum system. We compared the correlations quantified by linear relative entropy with ones obtained by means of Hilbert-Schmidt norm to understand the origin of deviation from the additivity property. To exemplify our analysis, we have considered a special class of two-qubit X states for which we obtained the analytical expressions of all types of correlations (classical, quantum and total) and the explicit form of their closest product states, closest classical states and closest classical product states.

References

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).

[2] V. Vedral, Rev. Mod. Phys. **74** (2002) 197.

[3] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**(2009) 865.

[4] O. Gühne and G. Tóth, Phys. Rep. **474** (2009) 1.

[5] L. Amico, R. Fazio, A. Osterloh and V. Vedral, Rev. Mod. Phys. **80** (2008) 517576.

[6] K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral, Rev. Mod. Phys. **84** (2012) 1655.

[7] A.K. Ekert, Phys. Rev. Lett. **67** (1991) 661.

[8] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.K. Wootters, Phys. Rev. Lett. **70** (1993) 1895.

[9] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69** (1992) 2881.

[10] C.A. Fuchs, Phys. Rev. Lett. **79** (1997) 1162.

[11] R. Rausschendorf and H. Briegel, `quant-ph/0010033`.

[12] D. Gottesman and I. Chuang, Nature **402** (1999) (6760) 390.

[13] P. Rungta, V. Buzek, C.M. Caves, M. Hillery, and G.J. Milburn, Phys. Rev. A **64** (2001) 042315.

[14] C.H. Bennett, D.P. DiVincenzo, J. Smolin, and W.K. Wootters, Phys. Rev. **A 54** (1996) 3824.

[15] W.K. Wootters, Phys. Rev. Lett. **80** (1998) 2245.

[16] V. Coffman, J. Kundu, and W.K. Wootters, Phys. Rev. A **61** (2000) 052306.

[17] L. Henderson and V. Vedral, J. Phys. A **34**(2001) 6899; V. Vedral, Phys. Rev. Lett. **90** (2003) 050401; J. Maziero, L. C. Celéri, R.M. Serra and V. Vedral, Phys. Rev A **80** (2009) 044102.

[18] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. **88** (2001) 017901.

[19] W.K. Wootters, Phys. Rev. Lett. **80** (1998) 2245; W.K. Wootters, Quant. Inf. Comp. **1** (2001) 27.

[20] S. Hill and W.K. Wootters, Phys. Rev. Lett. **78** (1997) 5022.

[21] S. Luo, Phys. Rev. A **77** (2008) 042303; Phys. Rev. A **77** (2008) 022301.

[22] M. Ali, A.R.P. Rau and G. Alber, Phys. Rev. A **81** (2010) 042105.

[23] M. Shi, W. Yang, F. Jiang and J. Du, J. Phys. A: Math. Theor. **44** (2011) 415304.

[24] M. Shi, F. Jiang, C. Sun and J. Du, New Journal of Physics **13** (2011) 073016.

[25] D. Girolami and G. Adesso, Phys. Rev. A **83** (2011) 052108.

[26] F.F. Fanchini,T. Werlang, C.A. Brasil, L.G.E. Arruda and A. O. Caldeira, Phys. Rev. A 81 (2010) 052107.

[27] P. Giorda and M. G. A. Paris, Phys. Rev. Lett. **105** (2010) 020503.

[28] G. Adesso and A. Datta, Phys. Rev. Lett. **105** (2010) 030501; G. Adesso and D. Girolami, Int. J. Quant. Info. **9** (2011) 1773.

[29] B. Dakic, V. Vedral and C. Brukner, phys. Rev. Lett. **105** (2010) 190502.

[30] J. Dajka, J. Luczka, and Peter Hänggi, Phys. Rev. A **84** (2011) 032120.

[31] J.D. Montealegre, F.M. Paula, A. Saguia, and M.S. Sarandy Phys. Rev. A **87** (2013) 042115.

[32] B. Aaronson, R. Lo Franco, G. Compagno and G. Adesso New Journal of Physics **15** (2013) 093022.

[33] F.M. Paula, T.R. de Oliveira, and M.S. Sarandy, Phys. Rev. A **87** (2013) 064101.

[34] K. Modi, T. Paterek, W. Son, V. Vedral and M. Williamson, Phys. Rev. Lett. **104** (2010) 080501.

[35] B. Bellomo, R. Lo Franco and G. Compagno, Phys. Rev. A 86, 012312 (2012).

[36] B. Bellomo, G.L. Giorgi, F. Galve, R. Lo Franco, G. Compagno and R. Zambrini, Phys. Rev. A **85** (2012) 032104.

[37] J. Briët and P. Harremoës, Phys. Rev. A **79** (2009) 052311.

[38] P.W. Lamberti, M. Portesi and J. Sparacino, Int. J. Quantum Inform. **07** (2009)1009.

[39] D. Girolami and G. Adesso, Phys. Rev. A **83** (2011) 052108.

[40] F.F. Fanchini,T. Werlang, C.A. Brasil, L.G.E. Arruda and A. O. Caldeira, Phys. Rev. A **81** (2010) 052107.

[41] F.F. Fanchini, M.F. Cornelio, M.C. de Oliveira and A.O. Caldeira, Phys. Rev. A **84** (2011) 012313.

[42] M. Daoud and R. Ahl Laamara, J. Phys. A: Math. Theor. **45** (2012) 325302.

[43] M. Daoud and R. Ahl Laamara, Int. J. Quantum Inform. **10** (2012) 1250060.