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REVIEW

Review of entangled coherent states

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Abstract

We review entangled coherent state research since its first implicit use in 1967 to the present. Entangled coherent states are important to quantum superselection principles, quantum information processing, quantum optics and mathematical physics. Despite their inherent fragility, entangled coherent states have been produced in a conditional propagating-wave quantum optics realization. Fundamentally the states are intriguing because they entangle the coherent states, which are in a sense the most classical of all states of a dynamical system.

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1. Introduction

Coherent states play an important role in representing quantum dynamics, particularly when the quantum evolution is close to classical. The coherent state was originally introduced by Schrödinger in 1926 as a Gaussian wavepacket to describe the evolution of a harmonic oscillator [1]. The centroid (mean values of the canonical variables) obtained from the Gaussian wavefunction follows the classical evolving harmonic oscillator, thereby mimicking its periodic evolution, and the spread of the wavepacket is fixed. Furthermore, the spread (variance) of this wavefunction satisfies the Heisenberg uncertainty relation and hence is as localized as possible within the constraints of quantum theory.

The coherent state emerged as an important representation [2] with the advent of the laser, and the concomitant desire to juxtapose quantum electrodynamics with analyses of coherent optical systems. As the electromagnetic field in free space can be regarded as a superposition of many classical modes, each one governed by the equations of a simple harmonic oscillator, the coherent state became significant as a tool for connecting quantum and classical optics.

The coherent state in quantum optics thus embodies the quantum-to-classical transition. Coherent states are minimum-uncertainty states. The centroid follows the evolution of the classical canonical variables in the classical optical description. In addition, coherent states are eigenstates of the annihilation operator, and hence correspond to classical noiseless fields in direct detection by ideal point electric-dipole detectors.

As coherent states are regarded as quasiclassical, the introduction of superpositions of coherent states rapidly became of widespread interest. Evidence of such superpositions first appeared in a study of a certain type of nonlinear Hamiltonian evolution (Hamiltonian commuting with number operator) by Milburn [3, 4], and the consequent manifestation of superpositions of coherent states was analyzed in detail by Yurke and Stoler [5, 6]. Studies of superpositions of coherent states for a single mode of the electromagnetic field were concerned with how to produce such states [7], their properties (such as squeezing, photon number distribution and robustness to environmental decoherence) and extensions to generalized coherent states [8–15].

Superpositions of coherent states have been reviewed by Bužek and Knight [16]. Superpositions of nearly distinct (i.e. weakly overlapping) coherent states earned the term 'cat state', in deference to Schrödinger's paradox of the cat, whose state of existence seems to be in a superposition of being dead versus alive [17]. As the states of death and life are considered to be macroscopically valid and distinct, the superposition of two coherent states, with large amplitude phases separated by π radians, is analogous to this paradox.

Superpositions of coherent states are difficult to produce, and fundamentally this could be due to extreme sensitivity to environmental decoherence. In fact this sensitivity is important in informing us as to why such peculiar states are not prevalent in nature. Experimental efforts to create cat states have concentrated on creating superpositions of coherent states that have limited distinguishability [18]. Such states have been dubbed 'Schrödinger kittens' [19].

Soon after the introduction of these single-mode superpositions of coherent states, entangled coherent states (or superpositions of multimode coherent states) became of widespread interest. Entanglement refers to the specific property that a state cannot be expressed as a convex sum of product states [20]. These superpositions of multimode coherent states arose independently in several papers. The earliest entangled coherent state appears in equation (10) of the 1967 Aharonov and Susskind analysis of charge superselection which shows that charges could appear in superposition [21]. Entangled coherent states thus provided a useful representation to explain charge superposition within the superselection paradigm [22].

The next appearance of the entangled coherent state appeared in equation (11) of Yurke and Stoler's 1986 seminal paper on generating superpositions of coherent states [5]. They suggested homodyne detection [23–25], which corresponds to a quadrature measurement [5, 6, 26], as a means for measuring the cat states. Homodyne detection is effected by mixing the cat state with a local oscillator in a coherent state with controllable phase. Mixing the local oscillator state with the cat state in an interferometer yields an entangled coherent state emerging as the output.

The pair coherent state [27–29], which is a special case of the Barut and Girardello $SO(2,1)\simeq SU(1,1)\simeq SL(2,\mathbb{R})$ coherent state [11], has an entangled coherent state representation in equation (2.6) of Agarwal's 1988 result [28]. Entangled coherent states as entities of physical interest in their own right first arose in a study by Tombesi and Mecozzi [30, 31], where the authors generalize the nonlinear birefringent evolution of Milburn [3, 4] and Yurke and Stoler [5] to multimode coherent states. Rather than employing the single-mode nonlinear evolution associated with the ideal optical Kerr nonlinearity, they treat the ideal Hamiltonian evolution of two orthogonally polarized light beams interacting in a nonlinear birefringent medium. After an appropriate evolution time, an initial two-mode coherent state evolves to an entangled coherent state [30–33].

Tombesi and Mecozzi then studied the relevant statistics of these states, such as photon number distribution and squeezing, as well as robustness to decoherence [31]. Agarwal and

Puri [34] studied the evolution of a two-mode coherent state through an optical Kerr medium and studied the entangled coherent state and its statistical properties. They also pointed out that their entangled coherent state is a simultaneous eigenstate of operators that are quadratic in the annihilation operators for the two modes.

The term 'entangled coherent state' was introduced by Sanders in a study concerning the production of entangled coherent states by using a nonlinear Mach–Zehnder interferometer [35, 36]. The nonlinear interferometer comprises a nonlinear medium in one path of a Mach–Zehnder interferometer. The nonlinear medium alone could suffice to produce entangled coherent states [30–33], but the interferometric set-up has analogies with the homodyne detection concept for superpositions of coherent states [5]. Linear optics alone is known to be insufficient to generate entangled coherent states so a nonlinearity is required [37].

Soon after the proposals to create entangled coherent states in two-mode propagating fields were made, a cavity quantum electrodynamics realization was proposed using one atom traversing two cavities and post-selecting on atomic measurement. This scheme was suggested for realizing entanglement between a coherent state in one mode and the vacuum in the other mode [38].

Entanglement of a coherent state with a vacuum state (which is a coherent state of zero amplitude) was a particular focus of the Sanders proposal [35]. In this analysis, a bipartite entangled coherent state was shown to violate a phase-coherence Bell inequality [39] in the few-photon limit [35, 36]. Later entangled coherent states were also shown to violate a formal Bell inequality in the large photon number limit [40].

Coherent states generated by a Kerr nonlinearity, within or without an interferometer, obey a conservation rule for total photon number, which constrains the phase relationship between the components of the multimode coherent state superposition. Chai [41] introduced entangled coherent states as superpositions of two-mode coherent states with equal amplitude, but opposite in optical phase, and allowed an arbitrary phase relationship between the two components of the bipartite superposition. The analysis then focused on the two-mode extension to single-mode even and odd coherent states [42]. Such states are sometimes called 'even entangled coherent states' and 'odd entangled coherent states' [43] and naturally generalize for q-coherent states [44, 45].

Chai studied statistical properties of even and odd entangled coherent states and showed that the joint photon number distribution of such entangled coherent states vanished for certain values of the photon number sum or difference. He also evaluated the squeezing properties of these even and odd entangled coherent states. These states also have quantum metrological applications [46, 47], and could be constructed by multimode parametric amplifiers [43]. Entangled coherent states also have applications to quantum information processing [48].

Although 'balanced', or equally weighted superpositions of multimode coherent states are typically studied, 'unbalanced', or unequally weighted superpositions are possible. An approximation to ideal unequally weighted superpositions can be generated by a nonlinear evolution within a double cavity system [49]. The requisite nonlinear evolution is actually a special case of the nonlinear evolution that leads to Titulaer–Glauber generalized coherent states [8–10].

Entangled coherent states were initially treated as bimodal states but later generalized to superpositions of multimode coherent states [50–52]. Generalizations to multimode systems allow the intricacies of multipartite entanglement to become manifest in entangled coherent states. Examples of interesting multipartite states having analogies in entangled coherent states include Greenberger–Horne–Zeilinger and W types of states [53, 54]. Another example is the cluster state [55–57].

As entangled coherent states exhibit entanglement, which is a resource for quantum information protocols, entangled coherent states have been studied both as a resource for, and also as an input to, a quantum information protocol. The degree of entanglement embodied by entangled coherent states was studied in the context of quantum information, where entanglement is considered a resource.

By showing that the even bipartite entangled coherent state can be obtained by mixing an even coherent state with the vacuum at a beam splitter, van Enk and Hirota [58] establish that the even entangled coherent state has precisely one ebit of entanglement, where one ebit is the amount of entanglement in a maximally entangled state of two qubits, or spin-$\frac{1}{2}$ particles. The degree of entanglement in a bipartite entangled coherent state generated by a nonlinear Kerr evolution was subsequently shown to yield an arbitrarily large amount of entanglement over proportionately short times and a limited amount of entanglement over longer times [59].

Entangled coherent states have been employed in quantum teleportation tasks [60] in two ways: as the state being teleported via continuous-variable quantum teleportation [61, 62] and as the entangled resource state employed to affect the teleportation [58, 61, 63, 64]. Teleportation gives operational meaning to the amount of entanglement in an entangled coherent state as teleportation consumes prior shared entanglement to transport quantum information through a classical channel [60].

Entangled coherent states can go beyond entangling harmonic oscillator coherent states. Earlier in this section we mentioned the Barut–Girardello coherent states [11], which can be used to construct pair coherent states, and these states are also an example of generalizing coherent states beyond the Heisenberg–Weyl algebra. Gilmore and Perelomov independently showed another way of generalizing coherent states based on abstracting the displacement operator to general group operations acting on minimum- or maximum-weight states [12–15]. The orbit of Gilmore–Perelomov states under the general group action forms the coherent states for the given group. For example, entangled coherent states can be constructed as superpositions of tensor products of two or more generalized Perelomov or Barut–Girardello $\mathfrak{su}(2)$ [65, 66] and $\mathfrak{su}(1,1)$ coherent states, as well as entangled binomial states [65].

In fact entangled coherent states arise naturally from the nonclassical coalgebraic structure of the generalized boson algebra $\mathcal{U}_{\langle q\rangle}(h(1))$ [67]. Squeezed states [68] are a generalization of coherent states as orbits of squeezed vacuum states under the Heisenberg–Weyl displacement operator. Specifically, squeezed states are constructed as orbits of the squeezed vacuum state, whereas coherent states are obtained by the same orbit construction but with the vacuum replacing the squeezed vacuum state as the fiducial state [26]. The topic of squeezing in the context of entangled coherent states arises when entangled coherent states are subject to squeezing [69]. Another example of the nexus between squeezed light and entangled coherent states arises when squeezed light is employed to enhance homodyne detection efficiency for superpositions of coherent states [31].

Generalization beyond the entangling of coherent states per se is also interesting. For example, coherent states can have photons added to them thereby creating ‘photon-added coherent states’ [29], which leads naturally to entangled photon-added entangled coherent states [70, 71]. The ‘single-mode excited entangled coherent states’ also involve adding a photon by acting on an entangled coherent state directly with a photon-creation operator [72, 73], and this state has value as a cyclic representation of the $\mathfrak{hw}$ algebra [72].

This article provides an overview of research into entangled coherent states and their generalizations, implementations and applications. This field of research is quite large so not every paper is cited, but this article strives to be comprehensive in covering all the directions concerning entangled coherent states. With the recent successful generation of entangled coherent states [74] and their potential importance in quantum information processing, many new discoveries can be expected in the near future.

2. Formalism

2.1. Coherent states of the simple harmonic oscillator

Coherent states of the simple harmonic oscillator are well known since the foundational work of Schrödinger [1] and the ubiquity of coherent states in quantum optics [75–77]. The Hamiltonian for the quantized simple harmonic oscillator in one dimension is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2, \tag{1}$$

with ˆ signifying an operator, m the mass of the oscillator, ω the angular frequency, and $\hat{q}$ and $\hat{p}$ the canonically conjugate Hermitian operators for position and momentum, respectively.

These conjugate operators satisfy the commutator relation $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$, and the Hamiltonian spectrum is $\left(n + \frac{1}{2}\right)\hbar\omega$ for n a non-negative integer. The harmonic oscillator thus has a ground state energy level of $\frac{1}{2}\hbar\omega$ and all excited energy levels are integer multiples of $\hbar\omega$ above the ground state energy level. For the simple harmonic oscillator, n indicates the number of quanta, with each additional quanta increasing the oscillator's mechanical energy by $\hbar\omega$.

As the energy levels are equally spaced, it is convenient to introduce the quanta lowering operator

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{\mathrm{i}\hat{p}}{\sqrt{2m\hbar\omega}} \tag{2}$$

and its conjugate raising operator $\hat{a}^\dagger$, which satisfy the commutator relation

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{1} \tag{3}$$

corresponding to the Heisenberg–Weyl Lie algebra $\mathfrak{hw}(1)$ comprising generators $\{\hat{a}, \hat{a}^\dagger, \mathbb{1}\}$ with $\mathbb{1}$ the identity operator. The Fock number states $|n\rangle$, for n the number of quanta, provide a countable, orthonormal basis for the Hilbert space $\mathscr{H}$, with n the number of quanta and $|n\rangle$ an eigenstate of the number operator:

$$\hat{n} \equiv \hat{a}^\dagger\hat{a}, \qquad \hat{n}|n\rangle = n|n\rangle. \tag{4}$$

The number states are energy eigenstates, hence are stationary states of the simple harmonic oscillator. An alternative is a Gaussian wavefunction (note that a Gaussian in position representation is also a Gaussian in the momentum representation, so it is sufficient to refer to a Gaussian wavefunction without specifying position or momentum representation), which satisfies the Schrödinger equation and is not stationary. The centroid (mean position and momentum) follows the simple harmonic motion expected for the classical simple harmonic oscillator, and the Gaussian remains a minimum-uncertainty state (with respect to uncertainty in position and momentum) so the Gaussian wavefunction has desirable properties.

This Gaussian, with initial conditions such that the position and momentum uncertainties are stationary, is known as a coherent state due to Glauber's association of such a state with coherence in quantum optics. Glauber's oscillating wavepacket is an eigenstate of the annihilation operator $\hat{a}$ for a given harmonic oscillator. This pure-state wavefunction is given in the Fock representation by

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{5}$$

with α a complex dimensionless amplitude such that the mean position and momentum are given by

$$\bar{q} = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha), \qquad \bar{p} = \sqrt{\frac{\hbar\omega}{2m}}\,\mathrm{Im}(\alpha), \tag{6}$$

respectively.

The quantity $\bar{n} = |\alpha|^2$ is the mean number of quanta, with the number of quanta governed by a Poisson distribution

$$\Pi_{\bar{n}} = |\langle n|\alpha = \sqrt{\bar{n}}\,\mathrm{e}^{\mathrm{i}\varphi}\rangle|^2 = \mathrm{e}^{-\bar{n}}\bar{n}^n/n! \tag{7}$$

and

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \tag{8}$$

the displacement operator. The Poisson distribution has the property that

$$\bar{n} = (\Delta n)^2, \tag{9}$$

and the Mandel Q parameter is [78]

$$Q = \frac{(\Delta n)^2 - \bar{n}}{\bar{n}}, \tag{10}$$

which is unity for the Poisson distribution (7), less than unity for sub-Poissonian distributions and greater than unity for super-Poissonian distributions.

For the extension to N simple harmonic oscillators, each indexed by an integer ℓ, the algebra for the identity $\mathbb{1}$ and the $2N$ operators is

$$\left\{\hat{a}_\ell, \hat{a}_\ell^\dagger; \ell = 1, 2, \ldots, N\right\}, \quad \left[\hat{a}_\ell, \hat{a}_{\ell'}^\dagger\right] = \delta_{\ell\ell'}\mathbb{1}. \tag{11}$$

The coherent state can be used for multiple harmonic oscillators. For N simple harmonic oscillators, the joint coherent state is

$$|\boldsymbol{\alpha}\rangle = \prod_{i=1}^{N} |\alpha_i\rangle = D(\boldsymbol{\alpha})|\mathbf{0}\rangle, \tag{12}$$

with $D(\boldsymbol{\alpha})$ a product of single-mode displacement operators (8) and $|\mathbf{0}\rangle$ the joint ground state of all N oscillators. As discussed above, a product coherent state with respect to a specific set of modes transforms to another product coherent state via a linear mode transformation.

Although this subsection has been concerned with the motion of the simple harmonic oscillator, and the energy quanta that separate the equally spaced energy levels have been referred to as quanta, the analysis also applies to the electromagnetic field. The free-space field is reducible to modes, and the dynamics of each mode corresponds to a simple harmonic oscillator. The energy quanta are photons as discussed in subsection 2.2.

2.2. Coherent states in quantum optics

In quantum optics, the electromagnetic field can be decomposed into modes, and the dynamics of each mode in free space is equivalent to the dynamics of the simple harmonic oscillator, with n the number of photons in the given mode. The canonically conjugate operators $\hat{q}$ and $\hat{p}$ are referred to as the in-phase and out-of-phase quadratures respectively, with the phase reference being a local oscillator.

As the phase of the local oscillator can be varied continuously, it is convenient in quantum optics to define the quadrature operator as

$$\hat{q}_\theta \equiv \hat{q}\cos\theta + \hat{p}\sin\theta, \tag{13}$$

with $\hat{q} = \hat{q}_0$ and $\hat{p} = \hat{q}_{\pi/2}$. Measurements of a quadrature are performed by mixing the signal field with the coherent local oscillator field in an optical homodyne detection apparatus [23–25], with the local oscillator field determining the phase θ.

In developing a theory of coherence for optical fields, Glauber employed the coherent state [75], but instead of the variables being the position and momentum of a massive particle in a harmonic potential, the canonically conjugate variables are the in-phase and out-of-phase quadratures of each mode of the field. The two quadrature field operators are constructed from the raising and lowering operators for the field mode with three-vector label $\vec{k}$ and polarization index ε, namely $\hat{a}^\dagger_{\vec{k}\varepsilon}$ and $\hat{a}_{\vec{k}\varepsilon}$ respectively. Properties of the coherent state have been discussed in subsection 2.1 and are investigated in detail by Klauder and Skagerstam [77].

Coherent states have served as a valuable tool for studying quantum optics, primarily because of the convenience of these states as representations. In addition to a coherent state being an eigenstate of the annihilation operator $\hat{a}$, another critical property is that the product coherent state is the unique state that transforms to a product coherent state under the action of a linear mode coupler such as a beam splitter [79]. For the mode coupler transformation given by

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ -\mathrm{e}^{-\mathrm{i}\phi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{14}$$

a product coherent state then transforms to a product coherent state. This property is particularly important in that a coherent state for the field remains a coherent state under any linear mode transformation.

Whether coherent states can be considered as ontologically real has been the subject of vigorous debate, both in the context of coherence of atomic Bose–Einstein condensates [80–82] and with respect to quantum teleportation of coherent states [22, 83–86]. The problem essentially concerns the establishment of the phase φ of the coherent state, either through its creation from a source or by a phase-sensitive detection of the state. In practice, phase-locking mechanisms exist that ensure that the phase of the coherent field is correlated with a reference field, and treating that field as classical, provides classical meaning to the parameter φ. The importance of this issue is noted in section 5 along with relevant references.

Given the indisputable value of the coherent state as a representation, there are two useful ways to represent the density matrix ρ of the single-mode field in terms of coherent states. One makes use of the Glauber–Sudarshan P-representation [76, 87] and the other makes use of the Husimi distribution, or Q function [88] (not to be confused with the Mandel Q [78]) whose advantages with respect to superpositions of coherent states were elaborated by Milburn [3].

The density matrix can be expressed in these representations as

$$\rho = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha), \qquad Q(\alpha) = \frac{\langle\alpha|\rho\,|\alpha\rangle}{\pi} \tag{15}$$

for $\mathrm{d}^2\alpha := \mathrm{d}\,\mathrm{Re}(\alpha)\,\mathrm{d}\,\mathrm{Im}(\alpha)$. In quantum optics, the field is said to be semiclassical if $P(\alpha)$ satisfies the axiomatic requirements of a probability density and is 'quantum' otherwise. In contrast, the Q-function is always a probability density.

There is a continuum of these so-called quasiprobabilities introduced by Cahill and Glauber [89], which are obtained through Gaussian convolution of the Glauber–Sudarshan $P(\alpha)$ representation, ranging from the P function to the Wigner function $W(q, p)$ to the Q function. The Wigner function is an especially important case because, although it is not positive definite, integrals of the Wigner function yield the marginal distributions for position and momentum. The generalized momentum marginal distribution is thus

$$P(q_\theta) = \int_{-\infty}^{\infty} W(q_\theta, q_{\theta+\pi/2})\, \mathrm{d}q_\theta \tag{16}$$

with q_θ a generalized canonical position parametrized by angle θ.

Titulaer and Glauber introduced 'generalized coherent states' as states that are fully coherent with respect to the coherence functions but are not eigenstates of the annihilation operator $\hat{a}$ [8]. These states have Poisson number distributions but allow an arbitrary phase relationship between coefficients in the Fock representation of the state. For $\boldsymbol{\vartheta} = (\vartheta_n)$ a vector of arbitrary phases,

$$\begin{aligned} |\sqrt{\bar{n}};\boldsymbol{\vartheta}\rangle &= \mathrm{e}^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}}{\sqrt{n!}}\mathrm{e}^{\mathrm{i}\vartheta_n}|n\rangle \\ &= \int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}|\sqrt{\bar{n}}\,\mathrm{e}^{\mathrm{i}\varphi}\rangle\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}(\vartheta_n - n\varphi)}, \end{aligned} \tag{17}$$

with the last line corresponding to the representation of the generalized coherent state as a superposition of coherent states on a circle. This representation on a circle was introduced by Białynicki–Birula [9].

Spiridinov [44] showed that these generalized coherent states are eigenstates of a generalized annihilation operator that holds the number operator $\hat{n}$ invariant. Physically Spiridonov's transformation corresponds to a number-sensitive rotation; optically we can understand this transformation as a generalization of the ideal single-mode optical Kerr nonlinearity, which affects a phase shift that is a function of field strength, or equivalently, photon number for the single-mode field.

2.3. Superpositions of coherent states

Whereas the coherent state is regarded as the closest quantum optical description to a classical coherent field, superpositions of coherent states exemplify the strangeness of quantum theory. In general any pure state of the field $|\psi\rangle$ can be written as a superposition of coherent states according to the expression

$$|\psi\rangle = \int \frac{\mathrm{d}^2\alpha}{\pi}\langle\alpha|\psi\rangle|\alpha\rangle. \tag{18}$$

As the coherent states form an overcomplete basis, it is not surprising that every state can be expressed as a superposition of coherent states.

Interestingly, the overcompleteness of the coherent-state basis allows quite different ways of writing the superposition. One particularly important case is the superposition of coherent states on the circle, which we have encountered in subsection 2.2, in studying the Titulaer–Glauber generalized coherent states [8]. Other states can also be expressed in this way. For example, the Fock number state has the appealing representation [9, 16, 90]

$$|n\rangle = \frac{1}{\sqrt{\Pi_n(m)}}\int\frac{\mathrm{d}\varphi}{2\pi}\,\mathrm{e}^{-\mathrm{i}m\varphi}|\sqrt{m}\,\mathrm{e}^{\mathrm{i}\varphi}\rangle, \tag{19}$$

with Π the Poisson distribution (7). Expression (19) is a superposition over coherent states with complex amplitude restricted to having modulus $\sqrt{m}$. States can also be expressed as superpositions of coherent states on lines or other subspaces of the α parameter space.

The evolution of a coherent state under an ideal optical Kerr nonlinearity

$$\vartheta(\hat{n}) = \omega\hat{n} + \lambda\omega^2\hat{n}^2 \tag{20}$$

yields a particular form of generalized coherent state [3, 4, 91]. Yurke and Stoler [5, 6] showed that a superposition of two coherent states could be obtained under this evolution, and its generalization to $\vartheta(\hat{n}) \propto \hat{n}^k$ could be expressed as a finite superposition of coherent states with different phases for certain evolution times. In fact the Titulaer–Glauber generalized coherent [8] can be expressed as a superposition of a finite number of coherent states on the circle for $\vartheta_{n+N} = \vartheta_n$ for some N and for all n [9, 10].

Spirodonov [44] identified two other interesting cases of generalized coherent states: q-deformed coherent states, for which $\vartheta_{n+N} = q\vartheta_n$, and parity coherent states $\vartheta_n = n\pi$. The parity operator is $\exp\{\mathrm{i}\pi\hat{n}\}$, and the (unnormalized) parity coherent state is given by

$$\mathrm{e}^{-\mathrm{i}\pi/4}|\alpha\rangle + \mathrm{e}^{\mathrm{i}\pi/4}|-\alpha\rangle, \tag{21}$$

which is a special case of the superpositions of coherent states with equal complex field amplitudes and equal phase separations studied by Białynicki-Birula [9] and Stoler [10].

Superpositions of coherent states on a circle can arise via the evolution of a coherent state according to a generalized Kerr nonlinearity, yielding an evolution operator $\exp\{\mathrm{i}\vartheta(\hat{n})\}$. The equally weighted superposition of two coherent states that are π out of phase with each other (21), introduced by Yurke and Stoler [5], has been termed a 'Schrödinger cat state', or 'cat state' for short, because the coherent state is regarded as being an essentially classical field state, and the superposition of two highly distinct coherent states is reminiscent of Schrödinger's cat being described as being in the state $|\text{'live'}\rangle + |\text{'dead'}\rangle$.

The term 'Schrödinger cat state' has also been applied to the 'even' and 'odd' coherent states [42],

$$|\alpha\rangle_\pm = N_\pm(|\alpha\rangle \pm |-\alpha\rangle) \tag{22}$$

for

$$N_+ = \frac{\exp(|\alpha|^2)}{2\sqrt{\cosh|\alpha|^2}}, \qquad N_- = \frac{\exp(|\alpha|^2)}{2\sqrt{\sinh|\alpha|^2}}. \tag{23}$$

The even–odd terminology refers to the fact that the photon number distribution is non-zero only for even photon number in the case of the even coherent state $|\alpha\rangle_+$ and is non-zero only for odd photon number in the case of the odd coherent state $|\alpha\rangle_-$. As this state does not have a Poisson number distribution, it cannot be evolved via a generalized unitary Kerr evolution from the coherent state but is a Titulaer–Glauber generalized coherent state [8] for which $\vartheta(\hat{n} + 2\mathbb{1}) = \vartheta(\hat{n})$. Even and odd coherent states may arise by a conditional Jaynes–Cummings evolution [38, 92].

Detection of Schrödinger cat states may be achieved by optical homodyne detection, with the measurement results converging to the marginal distributions for canonical position and momentum [5, 6] even in the presence of decoherence [30, 31]. Let us consider the 'balanced cat' of equation (21) with its equally weighted superposition of two coherent states π out of phase. If the local oscillator is in phase with either of the coherent states, the marginal distribution is equivalent to that for an incoherent mixture of such coherent states. The marginal distribution for the conjugate quadrature exhibits interference fringes that yield information on how coherent, or pure, the superposition state is.

So far we have considered superpositions of single-mode coherent states, but a superposition of multimode coherent states of type (12) is also allowed. Such a state can be written as

$$\int \mathrm{d}\mu(\boldsymbol{\alpha}) f(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle \tag{24}$$

for which the measure $\mathrm{d}\mu(\boldsymbol{\alpha})$ can be over the entire parameter space or over subspaces for which the set $\{\boldsymbol{\alpha}\}$ is an overcomplete basis. Before proceeding to studies of this superposition of multimode coherent states, we consider how coherent states and their superpositions are generalized to Lie groups and algebras other than the Heisenberg–Weyl group for simple harmonic oscillators.

2.4. Lie coherent states and their superpositions

The term 'generalized coherent state' has been used in subsection 2.3 to refer to a loosening of the phase relation between elements of the coherent state expressed as a superposition in the Fock basis; I refer to these as 'Titulaer–Glauber generalized coherent states' [8]. The term 'generalized coherent state' has also been applied to establishing coherent states for general Lie groups. Here I refer to Lie group and algebra generalizations of coherent states as 'Lie coherent states'. Where the specific Lie algebra is specified, the notation for the algebra replaces 'Lie', e.g. '$\mathfrak{su}(2)$ coherent state'. For N simple harmonic oscillators, the operator algebra is $\mathfrak{hw}(N)$ given in equation (11), and the Lie coherent state for $\mathfrak{hw}(N)$ is the multimode product coherent state (12).

Whereas $\mathfrak{hw}(N)$ coherent states are (i) displaced vacuum states (orbits of the vacuum state under the action of the displacement operator $D(\boldsymbol{\alpha})$), (ii) eigenstates of the lowering operator $\hat{\boldsymbol{a}}$ and (iii) minimum-uncertainty states, some sacrifices must be made in defining coherent states for other Lie groups. A basis set of operators for a Lie algebra can be expressed as lowering operators analogous to $\hat{a}$, the conjugate raising operators and the Cartan subalgebra, which is a set of mutually commuting elements of the algebra.

A Lie algebra generates a k-parameter Lie group with the dimension of the Cartan subalgebra being k. Studies of superpositions and entanglement of coherent states have so far focused primarily on one-parameter Lie groups (with the exception of one study on $\mathfrak{su}(3)$ coherent states [93]), so we restrict our attention to that case; in fact we can concentrate on $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$.

Entanglement of $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ coherent states was studied by Wang and Sanders [65]. The corresponding algebras are

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z, [\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm \tag{25}$$

and

$$[\hat{K}_+, \hat{K}_-] = -2\hat{K}_z, [\hat{K}_z, \hat{K}_\pm] = \pm\hat{K}_\pm, \tag{26}$$

respectively for $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$, with J used for the compact SU(2) group and K used for the non-compact SU(1,1) group. The Cartan subalgebras are J_z for $\mathfrak{su}(2)$ and K_z for $\mathfrak{su}(1,1)$. The Casimir invariants are $\hat{J}^2$ with spectrum $j(j+1)$ for irreducible representation, or irreps, indexed by $j \in \{0, 1/2, 1, 3/2, \dots\}$, and $\hat{K}^2$ with spectrum $k(k-1)$ for irreps indexed by

$$k \in \{1/2, 1, 3/2, 2, \dots\}. \tag{27}$$

Within a given irrep, an orthonormal basis is given by

$$\{|j\,m\rangle; m = -j, j+1, \dots, j\}, \tag{28}$$

such that

$$\hat{J}_-|j\,m\rangle = \sqrt{j-m+1}|j\,m-1\rangle \tag{29}$$

with $|j\ m-1\rangle \equiv 0$ if $m = -j$ and $\{|k\,n\rangle; n = 0, 1, 2, \dots\}$ for $\mathfrak{su}(2)$. The $\mathfrak{su}(1,1)$ can be constructed in a similar way.

For non-compact groups, there are two inequivalent ways to construct coherent states: (i) as eigenstates of the lowering operator and (ii) as orbits of a minimum uncertainty state, analogous to the orbit of the vacuum state under the displacement operator (8). Highest or lowest weight states (states that are annihilated by the raising and lowering operators, respectively) are typical choices of minimum-uncertainty states for the two groups under consideration.

In 1971, Barut and Girardello [11] introduced 'new coherent states' for non-compact groups based on criterion (ii). They identified the lowering operator(s) and found eigenstates for this operator. For $\mathfrak{su}(1,1)$, the lowering operator is $\hat{K}_-$, and the Barut–Girardello $\mathfrak{su}(1,1)$ coherent state is

$$|k\,\eta\rangle_{\mathrm{BG}} = \sqrt{\frac{|\eta|^{k-1/2}}{I_{2k-1}(2|\eta|)}}\sum_{n=0}^{\infty}\frac{\eta^n}{\sqrt{n!\Gamma(n+2k)}}|k\,n\rangle, \tag{30}$$

which satisfies $\hat{K}_-|k\,\eta\rangle_{\mathrm{BG}} = \eta|k\,\eta\rangle_{\mathrm{BG}}$, for $I_\nu(x)$ the modified Bessel function of the first kind.

For the case of the compact group $\mathfrak{su}(2)$, the usual minimum-uncertainty state is the highest-weight state $|j\,j\rangle$ although the lowest-weight state is used sometimes as well. Similarly, for $\mathfrak{su}(1,1)$, the lowest-weight state $|k\,0\rangle$ is used. For SU(2), the analog to the displacement operator is the 'rotation operator'

$$R(\theta,\varphi) = \left\{\frac{\theta}{2}[\mathrm{e}^{-\mathrm{i}\varphi}\hat{J}_- - \mathrm{e}^{\mathrm{i}\varphi}\hat{J}_+]\right\}, \tag{31}$$

and for SU(1,1), the analog is the 'squeeze operator' [94]

$$S(\xi) = \exp\{\xi\hat{K}_+ - \xi^*\hat{K}_-\}. \tag{32}$$

The term 'rotation operator' applies for SO(3), which is the rotation group in three-dimensional Euclidean space, and the term has been extended to apply to SU(2), which is a double covering group of SO(3). The term 'squeeze operator' is used here because the two-boson realizations of $\mathfrak{su}(1,1)$ are

$$\hat{K}_- = \hat{a}^2\;(k=1/4), \qquad \hat{K}_- = \hat{a}\hat{b}\;(k=3/4), \tag{33}$$

and either of these realizations of $\mathfrak{su}(1,1)$ converts the unitary operator (32) to the usual one-mode and two-mode squeeze operators in quantum optics for $k = 1/4$ and $k = 3/4$, respectively.

The $\mathfrak{su}(2)$ coherent states were first introduced as 'atomic coherent states' [12]. These states are given by

$$|j;\theta,\phi\rangle = R(\theta,\phi)|j\,j\rangle \tag{34}$$

and form an overcomplete basis of the Hilbert space [13, 14]. The coherent states are orbits of the minimum-uncertainty state under the action of a group element. Perelomov undertook a general analysis of such coherent states for any Lie group, and these Lie coherent states are known as Perelomov coherent states [15].

We refer to the Lie coherent state using the notation $|\ell\,\xi\rangle$ with ℓ the irrep parameter (not required for the $\mathfrak{hw}(n)$ algebra) and ξ the orbit parameter. This notation applies equally to eigenstates of the lowering operator (as for the Barut–Girardello states) and for orbits of minimum-uncertainty states (as for the Perelomov states). The multipartite Lie coherent state is designated by $|\ell\,\boldsymbol{\xi}\rangle$, which is a product of Lie coherent states $|\ell\,\xi_i\rangle$ all from the same irrep.

So far we have only considered entangled coherent states where each party in the state has the same coherent-state structure. For example, the entangled coherent state can be a superposition of a tensor product of $\mathfrak{hw}(1)$ coherent states or a superposition of tensor product of $\mathfrak{su}(2)$ coherent states or so on. On the other hand, a 'hybrid' entangled coherent state could be constructed as a superposition of tensor products of coherent states, with coherent states in the tensor-product space corresponding to different types of coherent states. Such hybrid coherent states have not been studied but have been realized experimentally in a limited way: hybrid $\mathfrak{hw}(1)$–$\mathfrak{su}(2)$ entangled coherent states are realized in cavity quantum electrodynamics experiments as entangled atom–field states [18, 38, 95].

2.5. Entangled coherent states

A superposition of mutimode or mutipartite coherent states can be expressed in general as [65]

$$\int \mathrm{d}\mu_\ell(\boldsymbol{\xi}) f_\ell(\boldsymbol{\xi}) |\ell\boldsymbol{\xi}\rangle \tag{35}$$

with the state $|\ell\boldsymbol{\xi}\rangle$ the Lie coherent state. For the usual case of harmonic oscillators, corresponding to the algebra $\mathfrak{hw}(N)$, the index ℓ is superfluous, and we let $\boldsymbol{\xi}$ be replaced by $\boldsymbol{\alpha}$ to obtain (24). This superposition is not entangled if there exists any representation for the pure state, such that the state can be expressed as a product state over the modes. Otherwise the state is entangled. Entangled coherent states are thus a special case of superpositions of multimode coherent states, but a rather large and especially interesting class of states.

The entangled state (35), which is expressed as an integral of product coherent states, can be reduced to a sum if the function $f_\ell(\boldsymbol{\xi})$ can be expressed as a sum of delta functions

$$f_\ell(\boldsymbol{\xi}) = \sum_i f_\ell(\boldsymbol{\xi}_i)\delta(\xi - \xi_i). \tag{36}$$

Then

$$\int \mathrm{d}\mu_\ell(\boldsymbol{\xi}) f_\ell(\boldsymbol{\xi}) |\ell\boldsymbol{\xi}\rangle = \sum_i f_\ell(\boldsymbol{\xi}_i) |\ell\boldsymbol{\xi}_i\rangle. \tag{37}$$

In the single-particle (equivalently single-mode) case, such states are the Titulaer–Glauber coherent states [8].

As an interesting example of discrete bipartite entangled coherent states, van Enk studied the discrete 'multidimensional entangled coherent states' [59]

$$|\Phi_M\rangle = \frac{1}{\sqrt{M}} \sum_{q=0}^{M-1} \mathrm{e}^{\mathrm{i}\phi_q} |\alpha\,\mathrm{e}^{2\pi\mathrm{i}q/M}, \alpha\,\mathrm{e}^{2\pi\mathrm{i}q/M}\rangle, \tag{38}$$

which are generated by an ideal nonlinear Kerr evolution (20), to characterize the entangling power. van Enk shows that such states have infinite entanglement after infinitesimally short evolution times τ and finite entanglement after finite (i.e. non-infinitesimal) times. Finite discrete superpositions can serve as a resource for quantum teleportation. Specifically van Enk demonstrated that, for very small losses for multidimensional entangled coherent states, approximately 2.89 ebits are lost per absorbed photon, which could be useful for creating entangled coherent states with a fixed amount of entanglement [96].

Entangled coherent states overlap conceptually with the pair coherent state, which was introduced as the joint eigenstate of the two-mode annihilation operator $\hat{a}_1\hat{a}_2$ and the number difference operator $\hat{n}_1 - \hat{n}_2$ [27, 28, 97]. The pair coherent state is defined by $|\zeta, q\rangle$, with ζ the eigenvalue of the pair annihilation operator and q the eigenvalue of the photon number difference operator. These states exhibit sub-Poissonian statistics, correlated number fluctuations, squeezing and violations of photon Cauchy–Schwarz inequalities.

Pair coherent states are an example of $\mathfrak{su}(1,1)$ coherent states, represented as two-boson realizations. The pair annihilation operator can be expressed according to algebra (26): the pair annihilation operator is $\hat{K}_-$ and the photon number difference operator is $\hat{K}_z$.

The pair coherent state is an example of an entangled coherent state, which is evident by expressing the pair coherent state as

$$|\zeta, q\rangle = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \frac{N_q\,\mathrm{e}^{|\zeta\rangle}}{[\sqrt{\zeta}\,\mathrm{e}^{\mathrm{i}\varphi}]^q} |\sqrt{\zeta}\,\mathrm{e}^{\mathrm{i}\varphi}\rangle |\sqrt{\zeta}\,\mathrm{e}^{-\mathrm{i}\varphi}\rangle \tag{39}$$

with the q-dependent normalization constant

$$N_q = \frac{|\zeta|^{q/2}}{\sqrt{I_q(2|\zeta|)}}, \tag{40}$$

for I_q the modified Bessel function of the first kind, as expressed by Gerry and Grobe [98].

The Schrödinger cat state concept, which corresponds to a superposition of coherent states, was extended to a superposition of pair coherent states by Gerry and Grobe [98]. Specifically the superposition of two pair coherent states can be expressed as

$$|\zeta, q, \phi\rangle = \frac{|\zeta, q\rangle + \mathrm{e}^{\mathrm{i}\phi}|-\zeta, q\rangle}{\sqrt{2 + 2N_q^2 \cos\phi \sum_{n=0}^{\infty} \frac{(-1)^n |\zeta|^{2n}}{n!(n+q)!}}}, \tag{41}$$

which is an eigenstate of the squared pair annihilation operator with eigenvalue $|\zeta|^2$. This superposition of pair coherent states is also an entangled coherent state, which is a superposition of two entangled coherent states of type (39).

3. Implementations

Entangled coherent states are fragile due to the fragility of their entanglement but are nonetheless implementable if the conditions are right. Furthermore, the states serve as a resource for quantum information processing so they have utility and value, hence are worth making. Many theoretical proposals exist for constructing entangled coherent states in the laboratory but so far the paramount experimental demonstration uses a photon-subtraction technique on two approximate Schrödinger cat states, so that the source of the photon is indeterminate [74, 99].

3.1. Parametric amplification and photodetection

Consider two physically separated states of light of the type $|\alpha\rangle + |-\alpha\rangle$. These two fields are each passed through a separate beam splitter so that each field loses a small fraction of its energy. The extracted part of each field is brought together after using a phase shifter to impose a ϕ phase difference between the two beams. The fields are combined at a beam splitter with a photon counter at one output port. The effect of this final beam splitter is to ensure that the detected photon is equally likely to have come from either beam. The resultant two-mode state conditioned on registering a single photon count is

$$-\mathrm{i}\sin\frac{\phi}{2}|\alpha\rangle|\alpha\rangle - \cos\frac{\phi}{2}|-\alpha\rangle|\alpha\rangle + \cos\frac{\phi}{2}|\alpha\rangle|-\alpha\rangle + \mathrm{i}\sin\frac{\phi}{2}|-\alpha\rangle|-\alpha\rangle. \tag{42}$$

This is the concept behind the successful experimental creation of a close approximation of this state, and the success of the process is verified by optical homodyne tomography on the resultant state [74].

The actual experiment involves using a pulsed optical parametric amplifier as a source of squeezed vacua. The cat state with small amplitude α, known as a 'kitten state', can be prepared by subtracting a single photon [19, 100]. Using this principle, the two output-mode squeezed vacua of orthogonal polarizations are recombined at a polarizing beam splitter. A small fraction of each field goes to the photon counter, which conditions the rest of the field going out of the other beam-splitter port into the entangled coherent state (42). The state is then tomographically characterized.

3.2. Nonlinear optics

The earliest proposals for creating entangled coherent states were expressed in the context of quantum optical fields interacting via a third-order optical nonlinearity known as a Kerr nonlinearity [101]. The optical Kerr nonlinearity features a refractive index $n_0 + n_2 I$, which is the sum of a linear refractive index n_0 and a second term that is proportional to the field strength, typically characterized by 'intensity' I. The Kerr effect, as a third-order optical nonlinearity, is a special case of four-wave mixing.

The term 'cross-Kerr nonlinearity' is ubiquitous in the entangled coherent state literature and refers to the phenomenon that one field experiences a phase-shift component that is proportional to the strength of the other field. The cross-Kerr effect thus leads to 'cross-phase modulation', which is specifically the phase shift of one field due to the intensity of the other. In the quantum analysis, the phase shift depends on the photon number in the other beam. Of course the phase shift of the beam also depends on its own strength, and this is known as 'self-phase modulation'.

The first proposal for generating entangled coherent states was introduced by Yurke and Stoler in 1986 [5, 6] followed by the work of Mecozzi and Tombesi in 1987 [30, 31] and then by others [32, 33]. The entangled coherent state became the chief object of interest in work by Sanders a few years later with a proposed implementation that inserted a Kerr nonlinearity into one path of a Mach–Zehnder interferometer (often called a 'nonlinear interferometer' for short) [35, 36] and has been the subject of further study [32, 33, 102, 103]. Related to this approach, if an appropriate superposition of two coherent states is provided, then a beam-splitter transformation alone suffices to produce an entangled coherent state from this resource [104].

A Kerr nonlinearity can be used to create an entangled coherent state with one of the two coherent states in the entangled state being in the vacuum state. Such an entangled coherent state was generalized by Luis to show how to create any bipartite entangled state with one of the two states being the vacuum state [105]. Wang showed how a nonlinearity coupled with linear optical elements can be employed to generate general bipartite entangled non-orthogonal states [106].

Variants of nonlinear interacting propagating-field realizations of entangled coherent states have been studied. Slow light in a medium with double electromagnetically induced transparency could be used to enable entangled coherent state generation [107, 108]. Entanglement could first be prepared in matter qubits and then transferred to fields to make entangled coherent states by exploiting a cross-Kerr nonlinearity [109]. Nonlocal preparation of a bipartite entangled coherent state, where 'nonlocal' means that the two fields being entangled never meet or directly interact, could be produced by sending a photon through a Mach–Zehnder interferometer with a nonlinear Kerr medium in each of its two paths, and separate coherent states could be sent through each of these two nonlinear media [110]. The bipartite entangled coherent state is post-selected by detecting from which port the photon leaves: whichever port the photon leaves from post-selects the nonlocal two-mode field in one of two entangled coherent states.

The generation of various exotic forms of entangled coherent states has been investigated. Greenberger–Horne–Zeilinger and W types of entangled coherent states could be produced with propagating fields using linear optics and Kerr nonlinearities [53, 54]. Similarly, cluster-type entangled coherent states can be generated with a nonlinear medium and a laser driving field [57, 111, 112].

3.3. Cavity quantum electrodynamics

Entangled coherent states can be created in cavity fields rather than in propagating fields, which has the advantage of large effective nonlinearities. The nonlinearity in the medium could be a macroscopic optical Kerr medium or one or more multilevel atoms. For example, a multimode entangled coherent state can be prepared by letting a single atom traverse two or more single-mode cavities, each occupied initially by a coherent state, and then post-selecting on the atomic state [38, 51, 56, 113–115]. An unbalanced (i.e. unequally weighted) entangled coherent state could be produced in a double-cavity system [49]. Alternatively just one cavity that supports a multimode field is an alternative to the multiple-cavity generation of entangled coherent states [113, 116–119].

Matter-wave interferometry could assist in preparing entangled coherent states. If a two-mode cavity can be prepared in a pair coherent state, then this state can be transformed into an entangled coherent state by the following procedure. Atoms are sent through a double slit and then interact with the two-mode cavity field. Atomic-position detection subsequently post-selects the two-mode field into an entangled coherent state [120].

Artificial atoms, such as quantum dots [121] or Cooper-pair boxes [122], can replace real atoms to produce entangled coherent states in cavity quantum electrodynamics. The microwave regime could prove to be quite appropriate for generating entangled coherent states with Rydberg atoms in millimeter-wave superconducting cavities [123, 124].

As with propagating fields interacting with a Kerr medium, exotic entangled coherent states, such as Greenberger–Horne–Zeilinger, W [125] and cluster-type entangled coherent states, can also be created in cavities [54, 56, 126–128]. Modified entangled coherent states, such as 'single-mode excited entangled coherent states', could also be created in a cavity quantum electrodynamics setting [72].

3.4. Motional degrees of freedom

Instead of creating entangled coherent states in electromagnetic field modes, motional degrees of freedom can be used instead. Vibrational degrees of freedom for a single trapped ion in two dimensions [129] or of two trapped ions [52, 130, 131], or for collective modes (e.g. center-of-mass or breathing modes) of many trapped ions [132, 133] can be transformed into entangled coherent states. Ion traps can be combined with cavity quantum electrodynamics set-ups to make hybrid entangled coherent states between electromagnetic and motional degrees of freedom [134, 135].

As an example of creating an entangled coherent state in the vibrational degrees of freedom of a single trapped ion in two dimensions, consider the interaction Hamiltonian [129]

$$\hat{H}_{\mathrm{Int}} = -\hbar\chi(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})(\hat{\sigma}_+ + \hat{\sigma}_-) \tag{43}$$

with $\hat{a}$ and $\hat{b}$ the annihilation operators for each of the two vibrational modes, χ a coupling coefficient and $\hat{\sigma}_+ = \hat{\sigma}_-^\dagger$ the electronic-energy lowering operator. Both vibrational modes are initialized in coherent states $|\alpha\rangle$ and $|\beta\rangle$ and the ion in the ground state $|\mathrm{g}\rangle$. At time t, the combined (unnormalized) state for the atom and the two vibrational states is

$$|\mathrm{g}\rangle(|\alpha\,\mathrm{e}^{\mathrm{i}\chi t}, \beta\,\mathrm{e}^{\mathrm{i}\chi t}\rangle + |\alpha\,\mathrm{e}^{-\mathrm{i}\chi t}, \beta\,\mathrm{e}^{-\mathrm{i}\chi t}\rangle) + |\mathrm{e}\rangle(|\alpha\,\mathrm{e}^{\mathrm{i}\chi t}, \beta\,\mathrm{e}^{\mathrm{i}\chi t}\rangle - |\alpha\,\mathrm{e}^{-\mathrm{i}\chi t}, \beta\,\mathrm{e}^{-\mathrm{i}\chi t}\rangle) \tag{44}$$

for $|\mathrm{e}\rangle$ the excited state of the atom. The entangled coherent state can be created post-selectively by measuring the electronic state of the ion.

Ion traps could be used to create multipartite entangled coherent states using entanglement swapping operations [52]. Consider two identical ions with each initially prepared in a superposition of ground and excited states and the center-of-mass and breathing modes each initially prepared in coherent states with the same amplitude and phase. Then two distinct Raman beams are directed at the two ions independently. One beam is directed at the first ion in order to couple it to the fundamental mode, and the second beam is directed at the second ion in order to couple it to the breathing mode.

This selective coupling of ion electron levels to motional modes is achieved by choosing judicious Raman parameters. Subsequently, Bell-state measurements of the two-ion electronic states result in the two motional modes 'collapsing', or being post-selected, into entangled coherent states. This principle is readily extended to the multimode entangled coherent state case by extending the number of ions from two to as many as desired, naturally accompanied by as many vibrational modes. The multi-ion electronic state is projected onto a maximally entangled state (generalized Bell measurement) thereby resulting in the vibrational modes being in a multimode (or 'multipartite' entangled coherent state) [52].

Other physical realizations that are amenable to creating entangled coherent states in motional degrees of freedom include nano-cantilevers [122] and movable nano-mirrors [136, 137]. More pointedly, entangled coherent states can in principle be realized in any system that can be described as harmonic oscillators with appropriate nonlinear coupling and sufficiently low loss and decoherence.

3.5. Bose–Einstein condensates

Bose–Einstein condensates have an inherently high nonlinearity due to atomic collision terms, and it is possible to prepare two separate Bose–Einstein condensates of three-level atoms (corresponding to different electronic states of the same atoms) into phase-locked coherent states, and couple them together via a Raman interaction [138]. This approach could be used to entangle two Bose–Einstein condensates with one of the two coherent states in the entangled coherent state being the ground state [105]. Alternatively entangled coherent states could be generated in Raman-coupled Bose–Einstein condensates [139].

Nonlocal preparation of distant entangled coherent states could be possible using electromagnetically induced transparency [140]. In this scheme, two strong coupling laser beams and two entangled probe laser beams prepare two distant Bose–Einstein condensates in electromagnetically induced, transparency-coherent population states, which are then forced to interact. The two Bose–Einstein condensates are initially in a product coherent state while the probe lasers are initially entangled. The final preparation step involves performing projective measurements upon the two outgoing probe lasers.

4. Nonclassical properties

Entangled coherent states are highly nonclassical states but are peculiar in that they are expressed as an entanglement of the most classically well-behaved states we know: coherent states. Thus, entangled coherent states are especially intriguing in studies of nonclassicality because the state represents an entanglement of classically meaningful descriptions of objects.

Nonclassicality is studied through a variety of measures including squeezing [33, 41, 141], sub-Poissonian photon statistics [41], violations of Cauchy–Schwarz inequalities [41], complementarity between particle-like and wave-like features of entangled coherent states [103], violations of the Bell inequality [35, 36, 40, 123, 124, 142, 143] or Leggett's inequality [144] for testing nonlocal realism, and entanglement properties such as index of correlation [65], entanglement of formation [145] and other measures [141]. Nonclassicality of generalized entangled coherent states, such as $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ states [65, 66] and photon-added entangled coherent states [71], has been studied as well.

4.1. Complementarity

Complementarity in double-slit [146] and two-channel interferometery [147] studies is well understood for single-particle inputs. Single photons exit wholly from either one or the other port of a beam splitter but exhibit strong fringe visibility when the experiment is modified by replacing the beam splitter by an interferometer [39]. Complementarity may be understood by thinking of the photon's path state as entangled: a superposition of the photon traversing one path (e.g. through one slit or down one channel of the interferometer) and a vacuum state in the other path and the reverse case. Now consider that, instead of a photon in one path and a vacuum in the other, we have a coherent state in one path and a vacuum in the other. Would complementarity be manifested and observable in that case?

Rice and Sanders showed that, in principle, a form of complementarity is present, but the notion of a phase shifter, which is a simple linear optical element for a photon, is complicated for a coherent state, yet necessary to observe the undularity of the coherent state in the context of entangled coherent states [103]. Joint photodetection at the two interferometer output ports [49] can reveal anticorrelation of the nonlinear Mach–Zehnder interferometer output, thereby revealing 'corpuscularity' of the coherent state analogous to the anticorrelation revealing corpuscularity for a single photon [103]. The coherent state is thus 'seen' to follow one path or another and not be split.

The ideal coherent state phase shifter would correspond to the unitary transformation $\exp(-\mathrm{i}\phi|\alpha\rangle\langle\alpha|)$ for the imposed phase shift ϕ and could be created in approximate form in a highly nonlinear medium with appropriate parameters [148]. The creation of this phase shifter would enable other types of tests of complementarity such as performing two-coherent-state interferometry, even with large numbers of photons. Two-coherent-state interferometry is analogous to two-particle quantum interferometry but with the single-particle Fock state replaced by a coherent state [148].

4.2. Entanglement

The nomenclature 'entangled coherent state' demands quantification of the degree of entanglement of such states. There is more than one way to study entanglement of such states. One can consider Bell inequalities or generalizations thereof, perhaps to test local realism or just to show non-factorizability. Another approach to studying entanglement of these states is to recognize that unentangled coherent states have Gaussian statistics and then use the covariance properties to quantify the degree of entanglement in such states [149]. An alternative approach considers the entangling power of operations that produce entangled coherent states [59]. Each of these approaches is challenging because the Hilbert spaces are infinite-dimensional and the entanglement is between non-orthogonal states [40, 106].

Quantifying entanglement can instead be studied in the context of performing a quantum information processing task such as quantum teleportation [60]. Teleportation enables a qubit to be sent from one party to another through a classical channel by sending instead two bits of information and consuming one 'ebit', or entangled bit (two maximally entangled qubits) of a prior shared entanglement resource.

Entanglement can then be quantified by determining how well entangled coherent states serve as the 'quantum channel' (i.e. the consumable prior shared resource) for teleporting another state. This other state could be a qubit corresponding to a superposition of single-mode coherent states ('a cat state') [58, 64, 104, 150–153]. The entangled coherent state can supply an entire ebit of resource despite being an entanglement of non-orthogonal states [58, 61, 63]. An alternative approach to studying quantum resources considers how well a given a resource serves to teleport all or part of an entangled coherent state [61, 62, 154–160].

Entanglement has been studied for various exotic forms of entangled coherent states. Both the Greenberger–Horne–Zeilinger type of entangled coherent states [53, 61, 151, 156, 161] and the W type of entangled coherent states [53, 161–163] have been studied as well as the cluster type of entangled coherent states [164].

The effect of dissipation and decoherence on entanglement and nonlocality is also the subject of intensive investigation for all types of entangled coherent states, including the robustness or fragility of the entanglement [165]. Probabilistic teleportation of coherent states via an entangled coherent-state quantum channel in an open system has been studied and characterized [145]. Non-Markovian decoherence dynamics is important for entangled coherent states, and An, Feng and Zhang obtain an exact master equation with and without environmental memory using influence-functional theory [166].

Strategies to mitigate decoherence of entangled coherent states, for example by squeezing [69], are of practical value. Entanglement purification for mixed entangled coherent states is also a promising approach [167, 168]. Park and Jeong compare the dynamics of entangled coherent states against entangled photon pair states under decoherence and inefficient detection [169]. They discover that entangled coherent states are more robust as quantum channels for teleportation, whereas entangled photon pair states are better with respect to photodetection inefficiency.

5. Applications and implications

Entangled coherent states have several applications as discussed earlier in this paper. For example, entangled coherent states can serve as a resource for quantum teleportation [58, 61, 63] or for quantum networks [170, 171]. A 'cat state' superposition of two coherent states readily serves as a qubit for quantum logical encoding [172].

The 'cat state' qubit also serves as the logical basis for performing universal quantum computation [173], and entangled coherent states play an important role in such quantum information processing [150]. In particular, this encoding leads to entangled qubits (ebits) corresponding to entangled coherent states [48].

Entangled coherent states also serve an important role in quantum metrology, which harnesses quantum resources such as entanglement to surpass the standard quantum limit (due to partition noise in particle interferometry, which applies to atomic clocks and displacement measurements *inter alia*) [94]. Multimode even/odd coherent states are especially amenable for quantum metrology [46]. Entangled coherent states are known to outperform other popular two-mode entangled states in quantum metrology [174–177], but perhaps their benefit is strongest for digital parameter discrimination [178].

The entangled coherent state representation [22] plays a key role in resolving fundamental issues concerning superselection of angular momentum [21], charge [21] and phase [22, 80–86]. Essentially, the entangled coherent state representation captures, in a mathematically simple and conceptually appealing way, namely how superselection can be obviated by adding an extra degree of freedom and splitting the state to provide a reference frame.

6. Summary and conclusions

This paper provides a comprehensive summary of results concerning entangled coherent states and their generalizations since the inception of entangled coherent states by Aharonov and Susskind in 1967 to obviate superselection. Coherent states are appealing for their mathematical elegance as representations and their closeness to classical physical states, and entangled coherent states build on these elegant representation properties.

Furthermore, entangled coherent states have a richness due to entanglement between these seemingly classical coherent states. Entangled coherent states have many beautiful nonclassical properties and generalize beyond the Heisenberg–Weyl algebra of harmonic oscillators to the cases of spin, squeezing, pair coherent states and beyond.

Remarkably, entangled coherent states have been created and observed experimentally. These exquisitely fragile states can be manifested in the laboratory given sufficient guile. Up to now the only successful experimental realization relies on parametric amplification in two modes and photon subtraction. Other realizations could be possible if large low-loss Kerr nonlinearities are created for propagation or for cavity fields. Ion traps could also be promising for realizing entangled coherent states between vibrational modes, and nanotechnology could open new vistas for entangling coherent states of motion.

Multipartite entanglement is a vast topic of research, and entangled coherent states play an important role in this area. Various multipartite entangled states such as Greenberger–Horne–Zeilinger, W and cluster states are studied for their rich properties and applications, and each of these states has nontrivial analogs with entangled coherent states.

In summary, entangled coherent states have been important from superselection arguments in 1967 to today's applications in quantum information processing. This review paper can serve as a resource to propel studies and applications of entangled coherent states in the coming decades, which hold further revelations and surprises.

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