

Unified scheme for correlations using linear relative entropy

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Abstract

A linearized variant of relative entropy is used to quantify in a unified scheme the different kinds of correlations in a bipartite quantum system. As illustration, we consider a two-qubit state with parity and exchange symmetries for which we determine the total, classical and quantum correlations. We also give the explicit expressions of its closest product state, closest classical state and the corresponding closest product state. A closed additive relation, involving the various correlations quantified by linear relative entropy, is derived.

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1 Introduction

Quantum entanglement in quantum systems, comprising two or more parts, constitutes a key concept to distinguish between quantum and classical correlations and subsequently to understand quantum-classical boundary. Besides its fundamental aspects, entanglement is commonly accepted of paramount importance in the development of quantum information science [1, 2, 3, 4, 5, 6]. In fact, entangled states have found various applications in quantum information processing protocols as for instance quantum cryptography [7], quantum teleportation [8], quantum dense coding [9]. Nowadays, entanglement is recognized as a valuable resource in several communication and computational tasks [10, 11, 12]. In view of these remarkable realizations and implementations, the concept of entanglement is expected to have many other implications and applications in others areas of research, especially condensed matter physics.

Therefore, the quantification and the characterization of quantum correlations between the components of a composite quantum system have attracted a special attention during the last two decades. The experimental and theoretical efforts, deployed in this context, are essential to develop the appropriate strategies to prevent against the decoherence effects induced by the system-environment coupling (see for instance the recent works [13, 14, 15] and references therein). Different measures were introduced from different perspectives and for various purposes [16, 17, 18, 19, 20, 21, 22]. Probably the most familiar among them is the quantum discord [23, 24] which goes beyond the entanglement of formation [25, 26]. It is given by the difference of total and classical correlations existing in a bipartite system. Now, it is well understood that almost all quantum states, including unentangled (separable) ones, possess quantum correlations. However, the analytical evaluation of quantum discord requires extremization procedures that can be tedious to achieve [27, 28, 29, 30, 31, 32, 33, 34]. To overcome this difficulty, a geometrical approach was proposed in [35]. It is based on the Hilbert-Schmidt norm in the space of density matrices. This measure provides explicit analytical expressions for pairwise quantum correlations. Clearly, Hilbert-Schmidt norm is not the unique distance which can be defined in the space of quantum states. Several distances are possible (trace distance, Bures distance, ...) with their own advantages and drawbacks and each one might be useful for some appropriate purpose [36, 37, 38, 39].

The states of any multipartite quantum system can be classified as being classical, quantum-classical and quantum states. Subsequently, the correlations can also be categorized in total, quantum, semi-classical (related to quantum-classical states) and classical correlations. This classification requires a specific measure (entropic or geometric distance) to decide about the dissimilarity between a given quantum state and its closest one without the desired property and to provide a consistent scheme to treat equally the different correlations. In this sense, using the relative entropy, an approach unifying the correlations in multipartite systems was recently developed in [40]. In particular, a very significant and interesting additivity relation was reported ($D + C = T + L$). It states that the sum of quantum D and classical C correlations is equal to the sum of total mutual correlations T and another

quantity L that is exactly the difference between D and the quantum discord as originally introduced in [23, 24].

However, it must be noticed that, despite its theoretical information meaning, the relative entropy is not symmetric in its arguments and therefore can not be considered as a true metric distance. In the other hand, from an analytical point of view, the derivation of closed expressions of relative entropy based measures involves optimization procedures that are in general very complicated to perform. In this respect, a purely geometrical unified framework to classify the correlations in a given quantum state was discussed in [41, 42]. Using the Hilbert-Schmidt norm and paralleling the definition of the geometric discord, the geometric measures of **total** and classical correlations in a two qubit system were derived in [41, 42]. In contrast with the relative entropy, the additivity relation of the type $(D + C = T + L)$ is not, in general, satisfied.

In this paper, we introduce a linearized variant of relative entropy. We obtain the explicit analytical expressions of quantum and classical correlations in a two qubit system. The relation with the geometric measure based on Hilbert-Schmidt norm is established. We show that the linear relative entropy provides us with a simple approach to treat the different kinds of bipartite correlations in a common framework. This approach can be seen, in some sense, interpolating between the **relative entropy-based** [23, 24] and Hilbert-Schmidt-based [41, 42] classification schemes. More specifically, it provides us with a very simple way to perform the optimizations required in deriving closest product, classical and classical product states. We also show that the correlations satisfy a closed additivity.

This paper is organized as follows. In section 2, we decompose the linear entropy in symmetric and anti-symmetric parts. We show that the antisymmetric part is related to quantum Jensen-Shannon divergence and the symmetric part is exactly the Hilbert-Schmidt distance [41]. Using the linear relative entropy, we obtain a closed additivity relation of the various bipartite correlations existing in a two qubit system. A comparison with Hilbert-Schmidt based approach is also investigated. As illustration, we consider, in section 3, a bipartite system possessing the parity symmetry and invariant under qubits permutation. **In this situation, the explicit derivations of the suitable closest product and classical states is achieved.** The analytical expressions of total, quantum and classical correlations are obtained and the additivity relation is discussed. Concluding remarks close this paper.

2 Correlation quantifiers based on symmetrized linear relative entropy

2.1 Correlation quantifiers **based on relative entropy**

The relative entropy offers the appropriate scheme to unify the different kinds of correlations existing in multipartite systems [40]. It is the quantum analogue of the Kullback-Leibler divergence between two classical probability distributions and characterizes the dissimilarity between two quantum states.

The relative entropy defined by

$$S(\rho\|\sigma) = -\text{Tr}(\rho \log \sigma) - S(\rho), \quad (1)$$

constitutes a quantitative tool to distinguish between the states of a given **degree of quantumness** and gives the distance between them according to the nature of their properties ($S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy). For a bipartite system, the total correlation $T = S(\rho\|\pi_\rho)$ is quantified by the relative entropy between a state ρ and its closest product state $\pi_\rho = \rho_A \otimes \rho_B$, where ρ_A and ρ_B denote the reduced **density** matrices of the subsystems. It writes as the difference of the von Neumann **entropies** [40]

$$T = S(\rho\|\pi_\rho) = S(\pi_\rho) - S(\rho). \quad (2)$$

Similarly, the quantum discord, which encompasses quantum correlations, is measured as the minimal distance between the state ρ and **its closest classical state**

$$\chi_\rho = \sum_{i,j} p_{i,j} |i\rangle\langle i| \otimes |j\rangle\langle j|, \quad (3)$$

where $p_{i,j}$ are the probabilities and $\{|i\rangle, |j\rangle\}$ local basis. It writes also as the difference between the von Neumann entropies of the states ρ and χ_ρ [40]

$$D = S(\rho\|\chi_\rho) = S(\chi_\rho) - S(\rho). \quad (4)$$

The classical correlation gives the distance between the **closest** classical state χ_ρ and its closest classical product state π_{χ_ρ} . It coincides with the difference of von Neumann entropies of the relevant states

$$C = S(\chi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\chi_\rho). \quad (5)$$

In this approach the **relative entropy-based** quantum correlations or quantum discord D (4) does not coincide with the original definition of discord introduced in [23, 24]. The difference is given by [40]

$$L = S(\pi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\pi_\rho). \quad (6)$$

The entropy-based correlations T , D , C and L are expressed as differences of von Neumann entropies (Eqs. (2), (4), (5) and (6)) and they satisfy the following remarkable additivity relation [40]

$$T - D - C + L = 0. \quad (7)$$

It must be emphasized that the relative entropy (1) is not symmetric under the exchange $\rho \leftrightarrow \sigma$. In this respect, it cannot define a distance from a purely mathematical point of view. Moreover, as mentioned above, the analytical evaluation of relative entropy-based correlations requires intractable minimization procedures. To avoid this problem, the linear relative entropy offers an alternative way to get computable expressions of correlations existing in multipartite systems [41].

2.2 Linear relative entropy

The linear entropy

$$S_2(\rho) \doteq 1 - \text{Tr}(\rho^2)$$

is related to the degree of purity, $P = \text{Tr}(\rho^2)$, and therefore reflects the mixedness in the state ρ . It is defined as a linearized variant of von Neumann entropy by approximating $\log \rho$ by $\rho - \mathbb{I}$ where \mathbb{I} stands for the identity matrix. Accordingly, the relative entropy (1) can be linearized as follows [41]

$$S_l(\rho_1 \parallel \rho_2) = \text{Tr} \rho_1 (\rho_1 - \rho_2). \quad (8)$$

It is not symmetric under the interchange of the states ρ_1 and ρ_2 . To define a symmetrized linear relative entropy, $S_l(\rho_1 \parallel \rho_2)$ is decomposed as the sum of two terms: symmetric and antisymmetric. The symmetric part is defined by

$$S_+(\rho_1 \parallel \rho_2) = S_l(\rho_1 \parallel \rho_2) + S_l(\rho_2 \parallel \rho_1). \quad (9)$$

The antisymmetric term is given by

$$S_-(\rho_1 \parallel \rho_2) = S_l(\rho_1 \parallel \rho_2) - S_l(\rho_2 \parallel \rho_1) \quad (10)$$

and rewrites as the differences between the linear entropies of the states ρ_1 and ρ_2

$$S_-(\rho_1 \parallel \rho_2) = S_2(\rho_2) - S_2(\rho_1). \quad (11)$$

We emphasize that the **antisymmetric** linear relative entropy (10) is related to the quantum Jensen-Shannon entropy of order 2 defined by

$$D_2(\rho_1 \parallel \rho_2) := S_2\left(\frac{\rho_1 + \rho_2}{2}\right) - \frac{1}{2}S_2(\rho_1) - \frac{1}{2}S_2(\rho_2). \quad (12)$$

Indeed, it can be expressed as

$$S_-(\rho_1 \parallel \rho_2) = D_2(\rho_1 + \rho_2 \parallel \rho_2 - \rho_1) - D_2(\rho_1 + \rho_2 \parallel \rho_1 - \rho_2). \quad (13)$$

The quantum Jensen-Shannon entropy of order 2 (12) is a **symmetrized form of the linear relative entropy**. It was recently used to investigate the distance between any two density operators (see for instance [43, 44] and references quoted therein) and subsequently constitutes an adequate geometric tool to classify quantum states according to their properties. The square root of quantum Jensen-Shannon divergence is a metric and can be isometrically embedded in a real Hilbert space equipped with a Hilbert-Schmidt norm [43]. This result is useful for our purpose. In fact, using (8) and (9), it is simple to check that the symmetric part of linear relative entropy is exactly the Hilbert-Schmidt distance [41]

$$S_+(\rho_1 \parallel \rho_2) = \|\rho_1 - \rho_2\|^2. \quad (14)$$

The symmetric (9) and antisymmetric (11) linear entropy are the main ingredients in this work. The **symmetric** linear relative entropy measures the distance between the states of a given quantum system

and the **antisymmetric** linear relative entropy quantifies the amount of correlations existing between two distinct states. With the linear relative entropy, considerable simplifications are possible in deriving the explicit expressions of total, quantum and classical pairwise correlations. Furthermore, interesting relations between the correlations (as measured by linear relative entropy) and their geometric counterparts (as defined in [41, 42] using the Hilbert-Schmidt distance) can be derived. This issue is discussed, in the next subsection, for an arbitrary two qubit system.

2.3 Additivity relation of geometric and entropic correlations

The Fano-Bloch representation of an arbitrary two-qubit state ρ is

$$\rho = \frac{1}{4} \sum_{\alpha, \beta} R_{\alpha\beta} \sigma_{\alpha} \otimes \sigma_{\beta} \quad (15)$$

where $\alpha, \beta = 0, 1, 2, 3$, $R_{i0} = \text{Tr}\rho(\sigma_i \otimes \sigma_0)$, $R_{0i} = \text{Tr}\rho(\sigma_0 \otimes \sigma_i)$ are components of local Bloch vectors and $R_{ij} = \text{Tr}\rho(\sigma_i \otimes \sigma_j)$ are components of the correlation tensor. **The operators σ_i ($i = 1, 2, 3$) stand for the three Pauli matrices and σ_0 is the identity matrix.** The distance (14), between two distinct density matrices ρ and ρ' , writes as

$$S_+(\rho\|\rho') \equiv d(\rho, \rho') = \frac{1}{4} \sum_{\alpha, \beta} (R_{\alpha\beta} - R'_{\alpha\beta})^2, \quad (16)$$

in terms of the Fano-Bloch coefficients $R_{\alpha\beta}$ (resp. $R'_{\alpha\beta}$) corresponding to ρ (resp. ρ'). The linear analogues of total correlation T (2), quantum correlation D (4), classical correlation C (5) and the quantity L (6) are respectively given by

$$T_2 = S_-(\rho\|\pi_{\rho}) \quad D_2 = S_-(\rho\|\chi_{\rho}) \quad C_2 = S_-(\chi_{\rho}\|\pi_{\chi_{\rho}}) \quad L_2 = S_-(\pi_{\rho}\|\pi_{\chi_{\rho}}). \quad (17)$$

Using the formula (11), it is easy to see that the correlations T_2 , D_2 , C_2 and L_2 can be written as differences of linear entropies. This implies the remarkable additivity relation

$$T_2 - D_2 - C_2 + L_2 = 0. \quad (18)$$

Inspired by the definition of the geometric discord based on Hilbert-Schmidt distance [35], Bellomo et al investigated a geometrical unified scheme in which the total correlation T (2), the quantum correlation D (4), the classical correlation C (5) and the quantity L (6) are redefined as [42]

$$T_g \equiv \|\rho - \pi_{\rho}\|^2, \quad C_g \equiv \|\chi_{\rho} - \pi_{\chi_{\rho}}\|^2, \quad D_g(\rho) = \|\rho - \chi_{\rho}\|^2, \quad L_g \equiv \|\pi_{\rho} - \pi_{\chi_{\rho}}\|^2. \quad (19)$$

It is worthwhile to mention that in view of the relation between the distance (9) and the Hilbert-Schmidt norm given by (14), **the linear relative entropy based correlations** can be expressed in terms of their geometric counterparts. Indeed, using (17), one shows

$$T_2 = T_g - 2S_2(\pi_{\rho}\|\rho), \quad D_2 = D_g - 2S_2(\chi_{\rho}\|\rho), \quad C_2 = C_g - 2S_2(\pi_{\chi_{\rho}}\|\chi_{\rho}), \quad L_2 = L_g - 2S_2(\pi_{\chi_{\rho}}\|\pi_{\rho}) \quad (20)$$

or alternatively

$$T_2 = T_g + 2\text{Tr}(\pi_\rho(\rho - \pi_\rho)), D_2 = D_g + 2\text{Tr}(\chi_\rho(\rho - \chi_\rho)), C_2 = C_g + 2\text{Tr}(\pi_{\chi_\rho}(\chi_\rho - \pi_{\chi_\rho})), L_2 = L_g - 2\text{Tr}(\pi_{\chi_\rho}(\pi_\rho - \pi_{\chi_\rho})) \quad (21)$$

Reporting the equations (20), or equivalently (21), in the additivity relation (18), one verifies that the geometric correlations (19) satisfy the relation

$$T_g - D_g - C_g + L_g = \Delta_g \quad (22)$$

where

$$\Delta_g = 2 \left[\text{Tr}(\pi_\rho(\pi_\rho - \rho)) + \text{Tr}(\pi_{\chi_\rho}(\chi_\rho - \pi_{\chi_\rho})) \right]. \quad (23)$$

We notice that the classical states χ_ρ (3) satisfy the relation $\text{Tr}\rho\chi_\rho = \text{Tr}\chi_\rho^2$ [35]. Hence, the geometric discord D_g coincides with the linear quantum correlation D_2 ($D_2 = D_g$). The geometric correlations satisfy a closed additivity relation similar to (18) when the quantity Δ_g (23) is zero. A detailed analysis of the relationship among the geometric correlations T_g , D_g , C_g and L_g for two qubit X states was recently investigated in [42]. In particular, it has been shown that the quantity Δ_g vanishes only in some special cases like for instance Bell states [41]. In this respect, to understand the differences between linear relative entropy and Hilbert-Schmidt based quantifiers, we shall consider a specific class of two qubit states for which one can explicitly evaluated total, quantum and classical pairwise correlations. This constitutes the main objective of the next section.

3 Analytical expressions of correlations

To illustrate the general results discussed in the previous section, we shall consider a family of two qubit density matrices whose entries are specified in terms of two real parameters. They are defined as

$$\rho = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1 c_2} \\ 0 & \frac{1}{2}(1 - c_1 - c_2) & \frac{1}{2}(1 - c_1 - c_2) & 0 \\ 0 & \frac{1}{2}(1 - c_1 - c_2) & \frac{1}{2}(1 - c_1 - c_2) & 0 \\ \sqrt{c_1 c_2} & 0 & 0 & c_2 \end{pmatrix} \quad (24)$$

in the computational basis $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The parameters c_1 and c_2 **satisfy the conditions $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and $0 \leq c_1 + c_2 \leq 1$** . We have taken all entries positives. In fact, the local unitary transformation

$$|0\rangle_k \rightarrow \exp\left(\frac{i}{2}(\theta_1 + (-)^k \theta_2)\right) |0\rangle_k$$

eliminates the phase factors of the off diagonal elements and the rank of the density matrix ρ remains unchanged. In the Fano-Bloch representation, the density ρ takes the form (15) and the corresponding (non-vanishing) correlation matrix elements are

$$R_{30} = R_{03} = c_1 - c_2 \quad R_{33} = 2(c_1 + c_2) - 1, \quad (25)$$

$$R_{11} = 1 - (\sqrt{c_1} - \sqrt{c_2})^2 \quad R_{22} = 1 - (\sqrt{c_1} + \sqrt{c_2})^2. \quad (26)$$

The density matrix (24) is invariant under parity symmetry and exchange transformation (ρ commutes with $\sigma_3 \otimes \sigma_3$ and the permutation operator which exchanges the qubit state $|i, j\rangle$ to $|j, i\rangle$ leaves ρ unchanged). These symmetries simplify considerably the complexity of the analytical evaluations of bipartite correlations.

3.1 Total correlation and closest product state

3.2 Closest product state

Let us derive the explicit expression of the total of total correlation T_2 defined by (17) in the bipartite state ρ (24). For this end, we first determine the closest product state to the density matrix ρ (24). An arbitrary product state writes

$$\pi_\rho = \rho_1 \otimes \rho_2 = \frac{1}{4} \left[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^3 (a_i \sigma_i \otimes \sigma_0 + b_i \sigma_0 \otimes \sigma_i) + \sum_{i,j=1}^3 a_i b_j \sigma_i \otimes \sigma_j \right] \quad (27)$$

where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ denote the unit Bloch vectors of the states ρ_1 and ρ_2

$$\rho_1 = \frac{1}{2} \left[\sigma_0 + \sum_{i=1}^3 a_i \sigma_i \right], \quad \rho_2 = \frac{1}{2} \left[\sigma_0 + \sum_{i=1}^3 b_i \sigma_i \right]. \quad (28)$$

Due to the symmetry of the state ρ under exchange of the qubits 1 and 2, the product state π_ρ is also symmetric. This implies

$$a_i = b_i \quad i = 1, 2, 3.$$

Furthermore, the parity symmetry of the density matrix ρ ($[\rho, \sigma_3 \otimes \sigma_3] = 0$) implies the parity invariance of the state π_ρ (27). This imposes

$$a_i = b_i = 0 \quad i = 1, 2.$$

Using the definition (16), the distance between the state ρ and π_ρ takes the simple form

$$d(\rho, \pi_\rho) = \frac{1}{4} [2(R_{30} - a_3)^2 + R_{11}^2 + R_{22}^2 + (R_{33} - a_3^2)^2] \quad (29)$$

to be optimized with respect one variable only, i.e. a_3 . It is clear that the parity and exchange symmetries simplify the minimization process in determining the closest product state. A minimal distance (29) is obtained for a_3 satisfying the following equation

$$a_3^3 + a_3(1 - R_{33}) - R_{30} = 0. \quad (30)$$

This cubic equation can be solved using Cardano's formula. Here, it is necessary to stress that for X states without exchange symmetry, the explicit determination of the Bloch coefficients a_i and b_i in (28) is complicated (see [42]). This explains why we deliberately chose to consider only two qubit states of type (24) which have both permutation and parity symmetries.

Being constrained to real solutions, the only real solution of (30) is

$$a_3 = \sqrt[3]{\frac{\sqrt{\Delta} + R_{30}}{2}} - \sqrt[3]{\frac{\sqrt{\Delta} - R_{30}}{2}} \quad (31)$$

where the discriminant Δ given by

$$\Delta = R_{30}^2 + \frac{4}{27}(1 - R_{33})^3$$

is always positive ($R_{33} \leq 1$). It follows that the closest product state to ρ (27) takes the form

$$\pi_\rho = \frac{1}{4} \left[\sigma_0 \otimes \sigma_0 + a_3 \sigma_3 \otimes \sigma_0 + a_3 \sigma_0 \otimes \sigma_3 + a_3^2 \sigma_3 \otimes \sigma_3 \right]. \quad (32)$$

It is interesting to note that for $c_1 = c_2$, the density matrix ρ (24) becomes a Bell-diagonal state. In this special case, the matrix elements R_{30} and R_{03} , given by (25), vanish and from (31) we have $a_3 = 0$. This implies that the closest product state (32) is just the product of the marginals $\rho_1 = \frac{1}{2}\sigma_0$ and $\rho_2 = \frac{1}{2}\sigma_0$ of ρ .

3.2.1 Total correlation

Using (17) together with (10), the total correlation writes

$$T_2 = \frac{1}{4} [2(R_{03}^2 - a_3^2) + R_{11}^2 + R_{22}^2 + (R_{33}^2 - a_3^4)]. \quad (33)$$

The behavior of total correlation T_2 versus c_1 is given in the figure 1 for different values of $\alpha = c_1 + c_2$ ($\alpha = 0.1, 0.2 \dots, 0.9$). In this figure, as well as in the other figures presented in this paper, the parameter c_1 vary from 0 to α . The minimal value of the total correlation is obtained for $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$. These two situations correspond respectively to states of the form

$$\rho(c_1 = 0, c_2 = \alpha) = \alpha |11\rangle\langle 11| + (1 - \alpha) |\psi_1\rangle\langle \psi_1|, \quad (34)$$

and

$$\rho(c_1 = \alpha, c_2 = 0) = \alpha |00\rangle\langle 00| + (1 - \alpha) |\psi_1\rangle\langle \psi_1|, \quad (35)$$

where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (36)$$

The total correlation T_2 is maximal for $c_1 = c_2 = \frac{\alpha}{2}$. In this case, the states of type (24) rewrite as

$$\rho(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}) = \alpha \rho_0 + (1 - \alpha) \rho_1 \quad (37)$$

where the states ρ_1 and ρ_0 are respectively given by

$$\rho_1 = |\psi_1\rangle\langle \psi_1| \quad (38)$$

where $|\psi_1\rangle$ is given by (36), and

$$\rho_0 = |\psi_0\rangle\langle \psi_0| \quad (39)$$

where $|\psi_0\rangle$ is the state defined by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

Furthermore, it can be clearly seen (figure 1) that when $0 \leq \alpha \leq 0.5$, the total correlation T_2 increases as the parameter α increases. For instance, for $c_1 = 0.05$ the amount of classical correlation, present in states (24) with $\alpha = 0.1$, exceeds ones measured by the linear entropy in the states with $\alpha = 0.2, 0.3, 0.4, 0.5$. The situation is completely different for $\alpha \geq 0.5$. Indeed, for small values of c_1 , the total correlation present in states with $\alpha = 0.6$ is higher than correlations exhibited by the states with $\alpha = 0.7, 0.8, 0.9$. For high values of c_1 ($c_1 = 0.55$ for instance), more correlation is contained in the states with $\alpha = 0.9$.

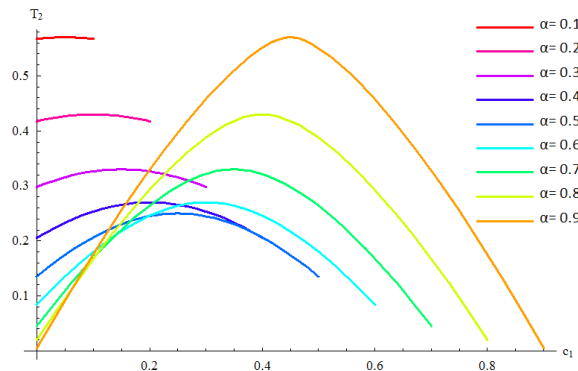


Figure 1. Total correlation T_2 versus the parameter c_1 for different values of $\alpha = c_1 + c_2$.

3.3 Quantum correlation and closest classical state

3.3.1 Quantum discord

The explicit computation of quantum discord, as originally defined in [23, 24], for an arbitrary bipartite system is difficult. Analytical results are known only in a few families of two-qubit states [27, 28, 29, 30, 45, 46, 47] (see also [48, 49] and references therein). The alternative geometric way, proposed in [35], quantifies the quantum discord as the minimal Hilbert-Schmidt distance between a given state ρ and the closest classical states of the form

$$\chi_\rho = \sum_{i=1,2} p_i |\psi_i\rangle \langle \psi_i| \otimes \rho_i \quad (40)$$

when the measurement is performed on the first subsystem. In equation (40), p_i is a probability distribution, ρ_i is the marginal density matrix of the second subsystem and $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an arbitrary orthonormal vector set. Following the procedure presented in [35], the explicit expression of the geometric quantum discord in the state (24) writes

$$D_g = \frac{1}{4} (R_{11}^2 + R_{22}^2 + R_{33}^2 + R_{03}^2 - \lambda_{\max}) \quad (41)$$

where the correlation elements are given by the expressions ((25)- (26)). In (41), the quantity λ_{\max} is defined by $\lambda_{\max} = \max (\lambda_1, \lambda_2, \lambda_3)$ where λ_1, λ_2 and λ_3 denote respectively the elements of the

diagonal matrix $K = \text{diag} (R_{11}^2, R_{22}^2, R_{33}^2 + R_{03}^2)$. As we already discussed , the quantum correlation D_2 coincides with the geometric measure of quantum discord. Thus, one gets

$$D_2 = D_g = \frac{1}{4} \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \quad (42)$$

The eigenvalues λ_1 , λ_2 and λ_3 , in terms of the parameters c_1 and c_2 , are

$$\begin{aligned} \lambda_1 &= [1 - (\sqrt{c_1} - \sqrt{c_2})^2]^2 \\ \lambda_2 &= [1 - (\sqrt{c_1} + \sqrt{c_2})^2]^2 \\ \lambda_3 &= \frac{1}{2} [(3c_1 + c_2 - 1)^2 + (c_1 + 3c_2 - 1)^2]. \end{aligned}$$

Noticing that λ_1 is always greater than λ_2 , we rewrite the geometric discord (42) as

$$D_g = \frac{1}{4} \min\{\lambda_1 + \lambda_2, \lambda_2 + \lambda_3\} = \frac{1}{4} (\min(\lambda_1, \lambda_3) + \lambda_2).$$

It is simple to verify that the difference $\lambda_3 - \lambda_1$ is positive when the parameters c_1 and c_2 satisfy the following condition

$$\sqrt{c_1}(2c_1 - 1) + \sqrt{c_2}(2c_2 - 1) \geq 0. \quad (43)$$

Otherwise, we have $\lambda_3 \leq \lambda_1$. Setting

$$\sqrt{c_1} = e^{-r} \cos \theta, \quad \sqrt{c_2} = e^{-r} \sin \theta \quad \text{with} \quad r \in \mathbb{R}, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

the condition (43) rewrites

$$e^{-r}(\cos \theta + \sin \theta)(2e^{-2r}(1 - \cos \theta \sin \theta) - 1) \geq 0.$$

This inequality is valid for the variables r and θ satisfying

$$2e^{-2r}(1 - \cos \theta \sin \theta) - 1 \geq 0$$

or equivalently

$$c_1 + c_2 - \sqrt{c_1 c_2} \geq \frac{1}{2} \quad (44)$$

in terms of the parameters c_1 and c_2 . The set of states of type (24) can be partitioned as

$$\{\rho \equiv \rho_{c_1, c_2}, \quad 0 \leq c_1 + c_2 \leq 1\} = \bigcup_{\alpha \in [0, 1]} \{\rho_\alpha \equiv \rho_{c_1, \alpha - c_1}, \quad 0 \leq c_1 \leq \alpha\}$$

with $c_1 + c_2 = \alpha$. The condition (44) is satisfied if and only if $\alpha \geq \frac{1}{2}$. Thus, for a fixed value $\alpha \leq \frac{1}{2}$, the quantity $\lambda_3 - \lambda_1$ is non positive and the geometric measure of quantum discord (42) writes

$$D_g = D_g^+ = \frac{1}{4} (\lambda_2 + \lambda_3). \quad (45)$$

For $\alpha \geq \frac{1}{2}$, the condition (44) is satisfied for

$$0 \leq c_1 \leq \alpha - \quad \alpha_+ \leq c_1 \leq \alpha$$

where

$$\alpha_{\pm} = \frac{\alpha}{2} \pm \frac{1}{2} \sqrt{(1-\alpha)(3\alpha-1)}. \quad (46)$$

In this case, the geometric quantum discord is given by

$$D_g = D_g^- = \frac{1}{4} (\lambda_1 + \lambda_2). \quad (47)$$

Conversely, for $c_1 \in [\alpha_-, \alpha_+]$ the difference $\lambda_3 - \lambda_1$ is negative and the geometric discord reads

$$D_g = D_g^+ = \frac{1}{4} (\lambda_2 + \lambda_3). \quad (48)$$

3.3.2 Closest classical state

Now we determine the explicit form of the closest classical state to the state (24). We consider separately the situations $\lambda_{\max} = \lambda_1$ and $\lambda_{\max} = \lambda_3$. We first treat the situation where $\lambda_1 \leq \lambda_3$. Following the general procedure developed in [35], we find that the zero discord (classically correlated) states are given by

$$\chi_{\rho}^- = \frac{1}{4} [\sigma_0 \otimes \sigma_0 + R_{30} \sigma_3 \otimes \sigma_0 + R_{30} \sigma_0 \otimes \sigma_3 + R_{33} \sigma_3 \otimes \sigma_3], \quad (49)$$

where the superscript $-$ refers to the condition $\lambda_1 - \lambda_3 \leq 0$. In this case, the pairwise quantum correlation writes

$$D_2^- = S_-(\rho \| \chi_{\rho}^-) = \frac{1}{4} (\lambda_1 + \lambda_2) = \frac{1}{4} (R_{11}^2 + R_{22}^2), \quad (50)$$

which can be reexpressed in terms of the parameters c_1 and c_2 as

$$D_2^- \equiv D_g^-(\rho) = \frac{1}{4} [1 - (\sqrt{c_1} - \sqrt{c_2})^2]^2 + \frac{1}{4} [1 - (\sqrt{c_1} + \sqrt{c_2})^2]^2.$$

It is interesting to note that the closest classical state χ_{ρ}^- satisfies

$$\text{Tr} \rho \chi_{\rho}^- = \text{Tr} \chi_{\rho}^{-2}$$

traducing that the geometric quantum discord D_g^- coincides indeed with the quantum correlation D_2^- . Along the same line of reasoning, one verifies that for $\lambda_1 > \lambda_3$, the closest classical state is given by

$$\chi_{\rho}^+ = \frac{1}{4} [\sigma_0 \otimes \sigma_0 + R_{03} \sigma_0 \otimes \sigma_3 + R_{11} \sigma_1 \otimes \sigma_1], \quad (51)$$

where the superscript $+$ refers now to the condition $\lambda_1 - \lambda_3 > 0$. In this case, the Hilbert-Schmidt distance between the density matrix ρ and its closest classical state χ_{ρ}^+ is

$$D_2^+ = S_-(\rho \| \chi_{\rho}^+) = \frac{1}{4} (\lambda_2 + \lambda_3) = \frac{1}{4} (R_{22}^2 + R_{03}^2 + R_{33}^2) \quad (52)$$

which rewrites also as

$$D_2^+ \equiv D_g^+(\rho) = \frac{1}{4} \left[[1 - (\sqrt{c_1} + \sqrt{c_2})^2]^2 + \frac{1}{2} [(3c_1 + c_2 - 1)^2 + (c_1 + 3c_2 - 1)^2] \right].$$

Here again, we notice that the closest classical state (51) satisfies the following relation

$$\text{Tr} \rho \chi_\rho^+ = \text{Tr} \chi_\rho^{+2}.$$

The behavior of the quantum discord D_2 , as function of the parameters c_1 and c_2 , is represented in the figures 2 and 3. Figure 2 gives the amount of quantum correlations for states with $\alpha \leq \frac{1}{2}$. As it can be inferred from figure 2, the quantum discord D_2 in the states (24) reaches its minimal value for $c_1 = \frac{\alpha}{2}$. These "minimally discordant" states are given by (37). Recall that the states of type (37) exhibit the maximal amount of total correlation T_2 (see figure 1). In the other hand, the maximal value of quantum discord D_2 is obtained in the states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ which are respectively given by the expressions (34) and (35). It is also interesting to note that in these "maximally discordant" states the total correlation T_2 is minimal (see figure 1). Thus, one concludes that for the states of type (24) with $c_1 + c_2 \leq \frac{1}{2}$, the quantum discord D_2 is maximal (resp. minimal) for states exhibiting a minimal (resp. maximal) amount of total correlation T_2 . The quantum discord D_2 and its first derivative evolves continuously (figure 1). This smooth behavior changes for states with $\alpha \geq \frac{1}{2}$ (figure 3). In fact, the quantum discord changes suddenly when $c_1 = \alpha_-$ and $c_1 = \alpha_+$ (α_- and α_+ are given by the expressions (46)). This sudden change of quantum discord occurs when the states ρ (24) have a maximum amount of quantum correlation. The behavior of quantum discord presents three distinct phases: $0 \leq c_1 \leq \alpha_-$, $\alpha_- \leq c_1 \leq \alpha_+$ and $\alpha_+ \leq c_1 \leq \alpha$. The minimal value of quantum discord D_2 is obtained in the intermediate phase ($\alpha_- \leq c_1 \leq \alpha_+$) for the states given by (37).

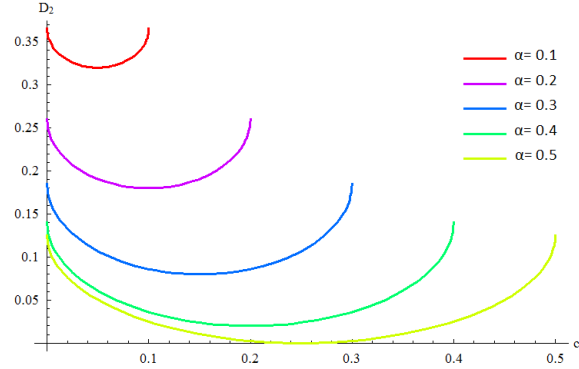


Figure 2. Quantum discord $D_2 \equiv D_g$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$.

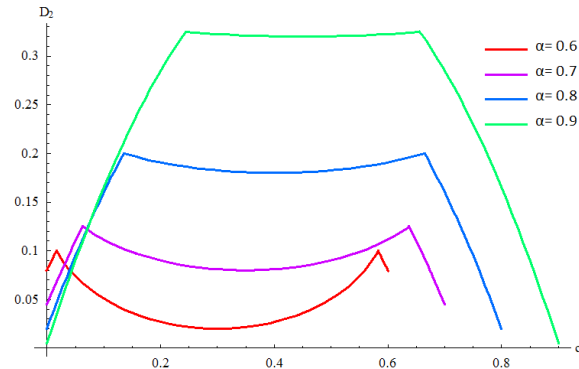


Figure 3. Quantum discord $D_2 \equiv D_g$ as function of the parameter c_1 for $\alpha \geq \frac{1}{2}$.

3.4 Classical correlations

The analytical derivation of classical correlations C_2 (17) requires the expressions of the closest product states to classical states χ_ρ^- (49) and χ_ρ^+ (51). We discuss first the situation where the classical state is given by χ_ρ^- (49). Remark that χ_ρ^- possesses parity and exchange symmetries. Thus, its closest product state can be obtained using the method leading to the closest product state (32). As result, one gets

$$\pi_{\chi_\rho^-} = \frac{1}{4} \left[\sigma_0 \otimes \sigma_0 + a_3 \sigma_3 \otimes \sigma_0 + a_3 \sigma_0 \otimes \sigma_3 + a_3^2 \sigma_3 \otimes \sigma_3 \right]. \quad (53)$$

which coincides with π_ρ . Subsequently, the classical correlation writes

$$C_2^- = \frac{1}{4} \left[2(R_{03}^2 - a_3^2) + (R_{33}^2 - a_3^4) \right]. \quad (54)$$

The determination of the closest classical product to classical state χ_ρ^+ (51) is slightly different from the previous case. In fact, the state χ_ρ^+ is invariant under parity transformation but not invariant under exchange symmetry. Accordingly, the closest classical product states are necessarily of the form

$$\pi_{\chi_\rho^+} = \frac{1}{4} \left[\sigma_0 \otimes \sigma_0 + \alpha_3 \sigma_3 \otimes \sigma_0 + \beta_3 \sigma_0 \otimes \sigma_3 + \alpha_3 \beta_3 \sigma_3 \otimes \sigma_3 \right], \quad (55)$$

where the variables α_3 and β_3 can be obtained by minimizing the Hilbert-Schmidt distance between the states χ_ρ^+ (51) and $\pi_{\chi_\rho^+}$ (55). After some algebra, one shows

$$\alpha_3 = 0 \quad \beta_3 = R_{03},$$

and the closest product state to χ_ρ^+ writes

$$\pi_{\chi_\rho^+} = \frac{1}{4} [\sigma_0 \otimes \sigma_0 + R_{03} \sigma_0 \otimes \sigma_3]. \quad (56)$$

Using the expressions (51), (56) and the definition (17), the classical correlation reads

$$C_2^+ = \frac{1}{4} R_{11}^2. \quad (57)$$

At this stage, we have the necessary ingredients to calculate explicitly the quantity L_2 defined by (17). Indeed, using the expressions of the closest product π_ρ (32) and the closest classical product states $\pi_{\chi_\rho^-}$ (53) and $\pi_{\chi_\rho^+}$ (56), one has

$$L_2^- = 0 \quad L_2^+ = \frac{1}{4} [2a_3^2 + a_3^4 - R_{03}^2], \quad (58)$$

and one recovers the additivity relation

$$T_2 + L_2^\pm = D_2^\pm + C_2^\pm$$

as expected. The figures 4 and 5 give the classical correlations C_2 as a function of the parameter c_1 for different values of $\alpha = c_1 + c_2$. For $\alpha \leq \frac{1}{2}$, the classical correlation C_2 behaves like the total

correlation T_2 depicted in figure 1. It is maximal for states satisfying $c_1 = c_2 = \frac{\alpha}{2}$ (37) and minimal for $(c_1 = 0, c_2 = \alpha)$ (Eq. (34)) and $(c_1 = \alpha, c_2 = 0)$ (Eq. (35)). Figure 5 shows a discontinuity of the classical correlations C_2 in the points $c_1 = \alpha_-$ and $c_1 = \alpha_+$ (Eq.(46)). These two particular values are exactly the points where the quantum discord changes suddenly. This discontinuity indicates that the quantity L_2^+ is non-vanishing and $C_2 \neq D_2^+ + C_2^+$ when the parameter c_1 ranges from α_- to α_+ . Remark that for $c_1 = c_2$, the density matrix ρ (24) is a Bell-diagonal state. In this case, we have $R_{03} = 0$ and $a_3 = 0$ which implies that the quantity L_2^+ is zero. This shows that in Bell-diagonal states, the total correlation T_2 is exactly the sum of quantum discord D_2 and classical correlation C_2 .

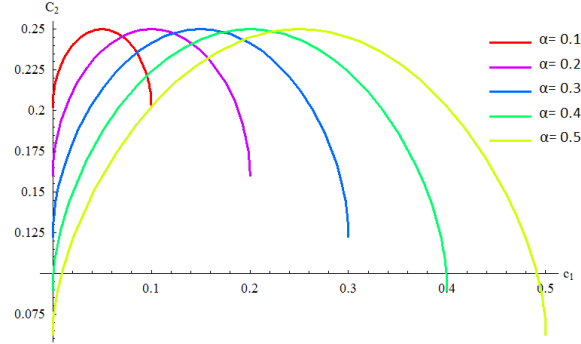


Figure 4. Classical correlations C_2 versus c_1 for $\alpha \leq \frac{1}{2}$.

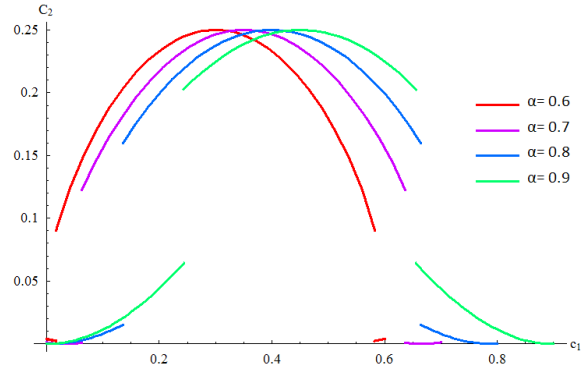


Figure 5. Classical correlations C_2 versus c_1 for $\alpha \geq \frac{1}{2}$.

3.5 Hilbert-Schmidt measures of correlations

The set of equations (20) establishes the relations between the correlation quantifiers based on relative entropy and geometric correlation quantifiers based on Hilbert-Schmidt distance. Indeed, using the expressions of closest product states π_ρ (32), one gets

$$\text{Tr}(\pi_\rho(\pi_\rho - \rho)) = \frac{1}{4}a_3^2(R_{33} - a_3^2). \quad (59)$$

Similarly, using the closest classical state χ_ρ^- (49) (resp. χ_ρ^+ (51)) and the closest classical product states $\pi_{\chi_\rho^-}$ (53) (resp. $\pi_{\chi_\rho^+}$ (56)), one shows

$$\text{Tr}(\pi_{\chi_\rho^-}(\chi_\rho^- - \pi_\rho)) = \frac{1}{4}a_3^2(a_3^2 - R_{33}) \quad (60)$$

and

$$\text{Tr}(\pi_{\chi_\rho^+}(\chi_\rho^+ - \pi_\rho)) = \frac{1}{4}R_{03}(R_{03} - a_3) \quad (61)$$

for $\lambda_1 \leq \lambda_3$ and $\lambda_1 > \lambda_3$ respectively. Reporting (59) in (21) and using the result (33), one obtains the geometric measure of the total correlation in the state ρ

$$T_g = \frac{1}{4} \left[2(R_{30} - a_3)^2 + R_{11}^2 + R_{22}^2 + (R_{33} - a_3^2)^2 \right]. \quad (62)$$

This result can be also derived from (19) using the expressions of the closest product state (32). Analogously, substituting (60) (resp. (61)) in C_g (21) and using the result (54) (resp. (57)), one gets

$$C_g^- = \frac{1}{4} (2(R_{30} - a_3)^2 + (R_{33} - a_3^2)^2) \quad (63)$$

and

$$C_g^+ = \frac{1}{4}\lambda_1 = \frac{1}{4}R_{11}^2. \quad (64)$$

Inserting the quantities L_2^\pm (58) in the relations involving L_2^\pm and L_g^\pm (21), one has

$$L_g^- = 0, \quad L_g^+ = \frac{a_3^2}{4} (1 + a_3^2 + a_3(a_3^2 - R_{33})^2). \quad (65)$$

Finally, using the equations (59), (60) and (61), the quantity Δ_g (23) is given by

$$\Delta_g^- = 0, \quad \Delta_g^+ = \frac{1}{2}a_3^2(R_{33} - a_3^2) \quad (66)$$

for $\lambda_1 \leq \lambda_3$ and $\lambda_1 > \lambda_3$, respectively. From the results (50), (62), (63), (65) and (66), one verifies that

$$T_g^- - D_g^- - C_g^- = 0$$

where T_g^- denotes the total correlation T_g in the states ρ depending on the parameters c_1 and c_2 satisfying the condition $\lambda_1 \leq \lambda_3$. In this case, the sum of the quantum correlation D_g^- and the classical correlation C_g^- coincides with the total correlation T_g^- . This result is no longer valid in the situation where $\lambda_3 < \lambda_1$. Indeed, from the equations (52), (62), (64), (65) and (66), we have

$$T_g^+ - D_g^+ - C_g^+ + L_g^+ = \Delta_g^+.$$

Using the equations (65) and (66), one verifies

$$\Delta_g^+ - L_g^+ = -\frac{1}{4}a_3^2 \left(a_3^2 + (a_3^2 + 1 - R_{33})^2 \right) \quad (67)$$

which implies that

$$T_g^+ - D_g^+ - C_g^+ \leq 0.$$

The last inequality becomes an equality for $a_3 = 0$. This solution is possible when $R_{30} = 0$ (see equation (31)). In this particular case, the state ρ (24) is a two-qubit state of Bell type. This agrees with the result derived in [42].

4 Concluding remarks

The concept of relative entropy provides a unified approach to treat equitably the different kinds of correlations existing in multipartite systems [40]. In this approach, total, quantum and classical correlations satisfy a closed additivity relation. However, the analytical use of the relative entropy is limited by the complex optimization procedures needed in minimizing the distance between a quantum state and its closest one without the required property. This constitutes the major drawback of the unified view based on the relative entropy. One is therefore forced to consider other measures. In this sense, Hilbert-Schmidt norm was adopted in [41, 42] to develop a purely geometric framework unifying the geometric variants of total, quantum and classical correlations. This approach has the advantage of working for a fairly general class of two qubit systems. But, the geometric correlations do not satisfy a closed additivity relation similar to one established in the relative entropy based approach. Motivated by these reasons, we formulated a scheme reconciling between the two above mentioned approaches. This scheme uses a linearized form of relative entropy and provides two major advantages. The first lies in its analytical simplicity to determine the different kinds of correlations. The second advantage concerns the additivity relation satisfied by the pairwise correlations. This unified approach has the added merit of being significantly simple in classifying the different states of bipartite systems according to their degrees of quantumness. We also explained how the scheme based on linear relative entropy can be used systematically to derive the pairwise geometric correlations based on Hilbert-Schmidt distance. To exemplify our analysis, we have considered a special class of two-qubit X states for which we obtained the analytical expressions of all pairwise correlations (classical, quantum and total) and the explicit form of their closest product states, closest classical states and closest classical product states.

Finally, we mention the general approach, recently proposed in [50], to remove the unexpected ambiguities leading to multi-valued quantum and classical correlations. Such ambiguities are essentially due to degeneracies arising from the optimization procedures of distance functions serving as correlations measures. In this context, we believe that it is worthwhile to examine the possible redefinitions of linear relative entropy quantifiers to avoid such problems. We hope to treat this issue in a forthcoming work.

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