Quantum correlation in multipartite photon added coherent states

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Abstract

We introduce the single-mode excited entangled coherent states and studied their quantum entanglement characteristics. Thus, we investigate the influence of photon excitations on quantum entanglement by using the different measurement (i.e. the concurrence, quantum discord and its geometric version). Therefor, To illustrate our results, we give the explicit expressions of the pairwise quantum correlations presented in multipartite coherent states and the special case also is analyzed. We found that the quantum discord and its geometrized variant does not follow the property of the concept of the monogamy, except is some particular situations that we presented.

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1 Introduction

The entanglement properties of strongly correlated quantum systems is presently attracting great attention in condensed matter physics [1],[2],[3],[4],[5]. Its understanding has led to the development of communication protocols like quantum teleportation [6],[7] and quantum cryptography [8],[9]. Entanglement is a purely quantum mechanical property of multipartite systems and is measurable in terms of nonclassical correlations of the subsystems. A system is entangled if its quantum state cannot be described as a direct product of states of its subsystems. Famous examples of entanglement are the Einstein-Podolsky-Rosen correlations between positions and momenta of two particles [10] and the violation of the Bell inequalities by spin systems that are described by Bell states, [11] which means that the existence of such states cannot be explained by any local theory. It has been known that there exist at least two different types of multipartite entanglement, namely, the GHZ-type [12] entanglement and the W-type entanglement [13]. However, most previous investigations have focused on the entanglement orthogonal state. Recent research shows that nonorthogonality plays an important role in quantum theory and quantum information such as quantum key distribution[14]. The bipartite entangled non-orthogonal states is studies in [15]. The entangled coherent state is one of the most important nonorthogonal states, and can be used to encode quantum information on continuous variables[16]. The even and odd Glauber coherent states, termed also *Shrödinger* cat states, can be considered as basis states of a logical qubit [17],[18] and provides a practical way to implement experimentally optical quantum systems useful for quantum information. On the other hand, a coherent state is the simplest continuous-variable state which is the closest analogue to a classical light field and exhibits a Poisson photon number distribution. Coherent states possess well defined amplitude and phase, whose uncertainties are the minimum permitted by the Heisenberg uncertainty principle. Based on coherent states, two types of continuous-variable states, called photon-added coherent states [19] and entangled coherent states [20],have been introduced and shown to have wide applications in both quantum physics [21] and quantum information processing [22],[23],[24],[25]. To study the distribution of quantum correlations in a quantum system among its different parts constitutes an important issue. In this scene monogamy relations are important, as they may indicate a structure for correlation in the multipartite setting [26],[27]. The concept of monogamy of entanglement was first proved by Coffman, Kundo and Wootters in 2001 [28] for three qubits and latter generalized for N qubits [29]. and since then it was extended to other measures of quantum correlations [30],[31], [32],[33], [34],[35].
In this work we consider a superposition of two and three-system coherent states, we focus our study to investigate the pairwise of quantum correlation for added tripartite coherent states, with two special measures. The entropic measure [36],[37],[38],[39],[40],[41],[42] and the geometric measure [43],[44],[45][46], as a function of the parameters involved. Our objective is study the dynamics of single-mode excited entangled coherent states.

This paper is organized as follows. In Section 2 we introduce the Glauber coherent states. We also present the form of the Single-mode excited entangled coherent states which are obtained by actions of creation operator and we study the evolution of the entanglement of Single-mode excited entangled coherent states. The explicit expression of three modes GHZ based on Glauber coherent states is derived in Section 3. Section 4 is devoted for study the evolution of the quantum correlations for each bipartite subsystems and we obtained the explicit form of quantum discord. The some special case is considered. Similar analysis are presented in Section 5 when the correlation are measured by means of the geometric discord. In section 6 we study the monogamy relation of the two measures, quantum discord and geometric discord. Finally, as illustration, some special cases are considered for the monogamy relation. Concluding remarks close this paper.

2 Entanglement of photon added coherent states

2.1 Photon added coherent states

We start this section with a brief description of the coherent states generated by a canonical annihilation and creation operators a and a^+ respectively. They satisfy canonical commutation relation $[a, a^+] = I$. The coherent states are generated by translating the vacum state $|0\rangle$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{1}$$

Where the operator of translation described by

$$D(\alpha) = exp(-\alpha a^+ - \alpha^* a). \tag{2}$$

Therefore the usual Glauber coherent states is given by the expression

$$|\alpha\rangle = \exp(\frac{-|\alpha|^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{3}$$

with $|\alpha|^2$ a complex dimensionless amplitude.
We consider m-photon excitations, by repeated application of the creation operator of m single mode $|\alpha\rangle$, which are satisfied the following form:

$$|\alpha; m\rangle = \left(a^+\right)^m \mid \alpha\rangle = \exp(\frac{-\mid \alpha \mid^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\sqrt{(n+m)!}\mid n+m\rangle, \tag{4}$$

the norm of the vector (4) is given by

$$\langle\alpha| \left(a^-\right)^m \left(a^+\right)^m |\alpha\rangle = \exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{\mid \alpha \mid^{2n}}{(n!)^2}(n+m)!, \tag{5}$$

we can easily obtain

$$\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle = \sum_{p=0}^{m}\frac{(m!)^2}{p!((m-p)!)^2}|\alpha|^{2(m-p)}, \tag{6}$$

finally $\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle$ and $\langle-\alpha|(a^-)^m(a^+)^m|\alpha\rangle$ can be written respectively by

$$
\begin{cases}
\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle & = & m!L_m(-|\alpha|^2), \\
\langle-\alpha|(a^-)^m(a^+)^m|\alpha\rangle & = & \exp(-2|\alpha|^2)m!L_m(|\alpha|^2),
\end{cases}
\tag{7}
$$

where $L_m(x)$ is the Laguerre polynomial of order m defined by

$$
L_m(x) = \sum_{n=0}^{m} \frac{(-1)^m m! x^n}{(n!)^2(m-n)!}. \tag{8}
$$

2.2 Bipartite quasi-Bell states

The quasi-Bell states are very interesting in quantum optics and have been used in the field of quantum teleportation and many others quantum computing operations. The Single-mode excited entangled bipartite quasi-Bell states are obtained through actions of a creation operator on coherent states, which are expressed by

$$
|\psi_\pm(\alpha, m)\rangle = N_\pm(\alpha, m)[(a^+)^m|\alpha\rangle \otimes |\alpha\rangle \pm (a^+)^m|-\alpha\rangle \otimes |-\alpha\rangle], \tag{9}
$$

one can directly obtain the factor of normalization as $N_\pm(\alpha, m)^{-2}$

$$
N_\pm(\alpha, m)^{-2} = 2m![L_m(-|\alpha|)^2 \pm \exp(-4|\alpha|^2)L_m(-|\alpha|)^2], \tag{10}
$$

now, we consider the following state

$$
|\psi_p(\alpha, m)\rangle = N_p(\alpha, m)[(a^+)^m|\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}(a^+)^m|-\alpha\rangle \otimes |-\alpha\rangle], \tag{11}
$$

where the integer p ($p \in \mathbb{Z}$). Thus, the state $\psi_p(\alpha, m)$ in terms of the photon-odded coherent states can be rewritten as

$$
|\psi_p(\alpha, m)\rangle = \mathcal{N}_p(\alpha, m)[|m;\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}|m;-\alpha\rangle \otimes |-\alpha\rangle], \tag{12}
$$

with

$$
\begin{cases}
\mathcal{N}_p(\alpha, m) & = & N_p(\alpha, m)\sqrt{\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle}, \\
|m;\pm\alpha\rangle & = & \frac{(a^+)^m|\pm\alpha\rangle}{\sqrt{m!L_m(-|\alpha|^2)}}.
\end{cases}
\tag{13}
$$

2.3 Entanglement of bipartite quasi-Bell states

In this subsection, we will calculate the entanglement of single-mode excited entangled coherent state in the orthogonal bases. Here we adopt the concurence to characterize the entanglement of the state $|\psi_p(\alpha, m)\rangle$ described by

$$
|\psi_p(\alpha, m)\rangle = \mathcal{N}_p(\alpha, m)[|m;\alpha\rangle \otimes |\alpha\rangle + e^{ip\pi}|m;-\alpha\rangle \otimes |-\alpha\rangle], \tag{14}
$$

we will introduce two qubits $|0\rangle$ and $|1\rangle$ for each subsystem

$$
\begin{matrix}
|m,\alpha\rangle = a_m|0\rangle + b_m|1\rangle; & \quad |m,-\alpha\rangle = a_m|0\rangle - b_m|1\rangle \\
|a|_m^2 + |b|_m^2 = 1; & \quad |a|_m^2 - |b|_m^2 = \langle m,\alpha|m,-\alpha\rangle
\end{matrix}
\tag{15}
$$

The overlapping function p_m and p are defined by

$$\begin{cases} p_m & = & \langle m, \alpha | m, -\alpha \rangle, \\ p & = & \langle \alpha | -\alpha \rangle, \end{cases} \tag{16}$$

and the two quantities a_m and b_m described in equation (15), are defined as

$$a_m = \sqrt{\frac{1+p_m}{2}}; \quad b_m = \sqrt{\frac{1-p_m}{2}}. \tag{17}$$

The state describing the systems is defined by

$$|\psi_p(\alpha, m)\rangle = \mathcal{N}_p(\alpha, m))[\underbrace{a_m a(1+e^{ip\pi})}_{C_{00}}|0,0\rangle + \underbrace{ab_m(1-e^{ip\pi})}_{C_{10}}|1,0\rangle + \underbrace{a_m b(1-e^{ip\pi})}_{C_{01}}|0,1\rangle + \underbrace{bb_m(1+e^{ip\pi})}_{C_{11}}|1,1\rangle]. \tag{18}$$

Following the approach of [23] and considering Eqs.(13), (17) and (18) for the excited bipartite entangled coherent states of photon excitations $|\psi_p(a, m)\rangle$, the concurrence can be calculated as

$$C = 2\mathcal{N}_p(\alpha, m)^2 |C_{00}C_{11} - C_{01}C_{10}|, \tag{19}$$

it is straightforward to check that the concurrence is given by

$$C = 2\mathcal{N}_p(\alpha, m)^2 \sqrt{1-p^2}\sqrt{1-p_m^2}, \tag{20}$$

where

$$p = \langle \alpha | -\alpha \rangle = \exp(-2|\alpha|^2), \tag{21}$$

and

$$p_m = \langle m, \alpha | m, -\alpha \rangle = \frac{\exp(-2|\alpha|^2) L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{22}$$

Then submitting equations (21) and (22) into (20), we see that

$$C(\alpha, m) = \frac{L_m(-|\alpha|^2)\sqrt{1-\exp(-4|\alpha|^2)}\sqrt{1-\exp(-4|\alpha|^2)(\frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)})^2}}{(L_m(-|\alpha|^2) + \exp(-4|\alpha|^2)\cos(k\pi) L_m(|\alpha|^2))}. \tag{23}$$

In order to observe the influence of the photon excitation on the quantum entanglement of a single mode excited bipartite entangled coherent states, we need to plot the concurrence C($|\alpha|$,m) versus $|\alpha|$ for different values of the number of excited photon m. Therefore, there are two case of k (even and odd) which are shown in figure 1, 2 respectively, we can see from the figure 1, after increasing of entanglement with $|\alpha|$, it approaches as the maximum value unit when $|\alpha|$ tends the infinity. In the other hand, the entanglement degrees increases as $|\alpha|$ during a number of photon increases. We can also observe in figure 2. That C($|\alpha|$,m) increases when $|\alpha|$ increase for differents values of m (m=0, 1, 2, 3, 4 and 10) respectively. Together the concurrence C($|\alpha|$,m) tends to unit for the larger $|\alpha|$.

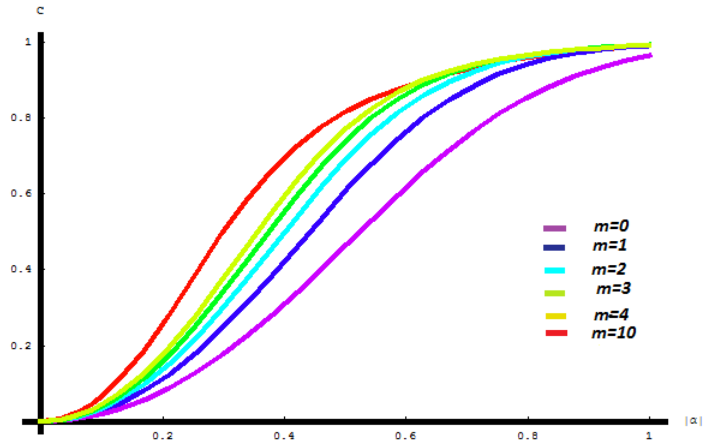

Figure 1: The concurrence C (even) of SMEECS $|\psi(|\alpha|, m)\rangle$ versus $|\alpha|$ for the different number photon excitations.

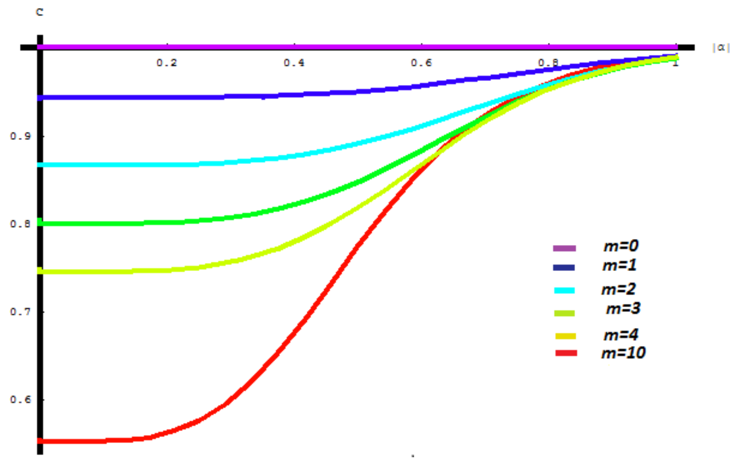

Figure 2: The concurrence C (odd) of SMEECS $|\psi(|\alpha|, m)\rangle$ versus $|\alpha|$ for the different number photon excitations.

3 Single mode excited Tripartite GHZ coherent states

In this section, we are interested to study the tripartite quantum correlations present in single mode excited entangled coherent states (27) and discuss their mathematical properties.
We recall that the three-mode GHZ coherent states are expressed as

$$|\alpha, 0\rangle = \mathcal{N}_0(|\alpha, \alpha, \alpha\rangle + exp(ik\pi)| - \alpha, -\alpha, -\alpha\rangle). \tag{24}$$

Where the normalization constant $\mathcal{N}_0$ is given by

$$\mathcal{N}_0 = (2 + 2p^3 \cos(k\pi))^{-\frac{1}{2}}, \tag{25}$$

and p is the overlap between two states

$$p = \langle\alpha| - \alpha\rangle = \exp(-2|\alpha|^2). \tag{26}$$

We consider m-photon excitations of the mod-1

$$|\alpha, m\rangle = \mathcal{N}a^{+m}(|\alpha,\alpha,\alpha\rangle + exp(ik\pi)|-\alpha,-\alpha,-\alpha\rangle) = \frac{\mathcal{N}}{\mathcal{N}_0}a^{+m}|\alpha,0\rangle, \tag{27}$$

from the norm of this state $\langle\alpha, m|\alpha, m\rangle = 1$, one can easily find that the normalization factor $\mathcal{N}$ is given

$$\mathcal{N} = [2m!(L_m(-|\alpha|^2) + p^3 cosk\pi(L_m|\alpha|^2)]^{\frac{-1}{2}}. \tag{28}$$

In general for reducing the total system described by the matrix ρ_{ij} to its sub-system, is performed by the trace on all other sub-systems. It is easy to see that reduced density matrices ρ_{12}, ρ_{13} are identical. Then, by tracing out systems 1 in the state $|\alpha, m\rangle$, we obtain the reduced density matrix ρ_{23} as follows

$$\rho_{23} = tr_1\rho_{123}, \tag{29}$$

Where

$$\rho_{123} = |\alpha, m\rangle\langle\alpha, m|, \tag{30}$$

in this follows case, we note

$$a^{+m}|\alpha\rangle \equiv |\alpha; m\rangle; \qquad a^{+m}|-\alpha\rangle \equiv |-\alpha; m\rangle. \tag{31}$$

Indeed, the reduced density matrice ρ_{12} take this form

$$\begin{aligned}\rho_{12} &= \mathcal{N}^2(|\alpha; m,\alpha\rangle\langle\alpha; m,\alpha| + |-\alpha; m,-\alpha\rangle\langle-\alpha; m,-\alpha| \\ &+ e^{-ik\pi}p|\alpha; m,\alpha\rangle\langle-\alpha; m,-\alpha| + e^{ik\pi}p|-\alpha; m,-\alpha\rangle\langle\alpha; m,\alpha|),\end{aligned} \tag{32}$$

in the same way, we obtain that

$$\rho_{12} = \rho_{13}. \tag{33}$$

Thus, we get

$$\begin{aligned}\rho_{12} &= \mathcal{N}^2[|\alpha,\alpha\rangle\langle\alpha,\alpha|\langle\alpha|a^{-m}a^{+m}|\alpha\rangle + |-\alpha,-\alpha\rangle\langle-\alpha,-\alpha|\langle-\alpha|a^{-m}a^{+m}|-\alpha\rangle \\ &+ |\alpha,\alpha\rangle\langle-\alpha,-\alpha|e^{-ik\pi}\langle-\alpha|a^{-m}a^{+m}|\alpha\rangle + |-\alpha,-\alpha\rangle\langle\alpha,\alpha|e^{ik\pi}\langle\alpha|a^{-m}a^{+m}|-\alpha\rangle],\end{aligned} \tag{34}$$

note that

$$\langle\alpha|a^{-m}a^{+m}|\alpha\rangle = \langle-\alpha|a^{-m}a^{+m}|-\alpha\rangle = m!L_m|\alpha|^2, \tag{35}$$

and

$$\langle-\alpha|a^{-m}a^{+m}|\alpha\rangle = pm!L_m|\alpha|^2, \tag{36}$$

concerning the reduced density matrix $\rho_{2,3}$ can easily obtain as follows

$$\begin{aligned}\rho_{2,3} &= \mathcal{N}^2[|\alpha,\alpha\rangle\langle\alpha,\alpha|m!L_m(-|\alpha|^2) + |-\alpha,-\alpha\rangle\langle-\alpha,-\alpha|m!L_m(-|\alpha|^2) \\ &+ |\alpha,\alpha\rangle\langle-\alpha,-\alpha|e^{-ik\pi}pm!L_m|\alpha|^2 + |-\alpha,-\alpha\rangle\langle\alpha,\alpha|e^{ik\pi}pm!L_m|\alpha|^2],\end{aligned} \tag{37}$$

which can be written also as

$$\begin{aligned}\rho_{2,3} &= \mathcal{N}^2 m! L_m(-|\alpha|^2)[|\alpha,\alpha\rangle\langle\alpha,\alpha| + |-\alpha,-\alpha\rangle\langle-\alpha,-\alpha| \\ &+ \frac{pL_m|\alpha|^2}{L_m(-|\alpha|^2)}|\alpha,\alpha\rangle\langle-\alpha,-\alpha|e^{-ik\pi} + |-\alpha,-\alpha\rangle\langle\alpha,\alpha|e^{ik\pi}].\end{aligned} \tag{38}$$

To study the tripartite quantum correlations present in eq.(27), a new formalism is used to describe our system

$$|\psi\rangle_{123} = \mathcal{N}^{'}(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle + e^{im\pi}|\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle), \tag{39}$$

the constant of normalization $\mathcal{N}^{'}$ can be written in the form

$$\mathcal{N}^{'} = (2 + 2cosp_1p_2p_3cos(m\pi))^{-\frac{1}{2}}, \tag{40}$$

recall that the general form of reduced density matrix in the bipartite system can be written as

$$\rho_{ij} = \mathcal{N}^2[|\psi_i\psi_j\rangle\langle\psi_i\psi_j| + |\phi_i\phi_j\rangle\langle\phi_i\phi_j| + e^{ik\pi}q_{ij}|\phi_i\phi_j\rangle\langle\psi_i\psi_j| + e^{-ik\pi}q_{ij}|\psi_i\psi_j\rangle\langle\phi_i\phi_j|]. \tag{41}$$

The quantity q_{ij} mentioned in the eq. (41) is defined by

$$q_{ij} = p_1p_2...\breve{p}_i...\breve{p}_j...p_n, \tag{42}$$

one can see that the density ρ_{ij} take the following compact form

$$\rho_{ij} = \frac{\mathcal{N}^2}{\mathcal{N}_{ij}^2}[a_{ij}^2|\psi_{ij}\rangle\langle\psi_{ij}| + b_{ij}^2 Z|\psi_{ij}\rangle\langle\psi_{ij}|Z], \tag{43}$$

with

$$|\psi_{ij}\rangle = \mathcal{N}_{ij}(|\psi_i,\psi_j\rangle + +e^{im\pi}|\phi_i\phi_j\rangle, \tag{44}$$

$|\psi_i\rangle$ and $|\phi_i\rangle$ is written as

$$|\psi_i\rangle = a_i(|0_i\rangle + b_{_i}|b_i\rangle, \tag{45}$$

$$|\phi_i\rangle = a_i(|0_i\rangle + -b_{_i}|b_i\rangle, \tag{46}$$

where

$$a_i = \sqrt{\frac{1+p_i}{2}}; \qquad b_i = \sqrt{\frac{1-p_i}{2}}. \tag{47}$$

Similarly for the subsystem j.

The state ψ_{12} is defined by the following expression:

$$|\psi_{12}\rangle = \mathcal{N}_{12}(|\alpha;m\rangle|\alpha\rangle + e^{im\pi}|-\alpha;m\rangle|-\alpha\rangle), \tag{48}$$

for the subsystem 1

$$|\alpha;m\rangle = a_1(|0\rangle_1 + b_{_1}|1\rangle_1; \qquad |-\alpha;m\rangle = a_1(|0\rangle_1 - b_{_1}|1\rangle_1, \tag{49}$$

with

$$a_1 = \sqrt{\frac{1+p_1}{2}}; \qquad b_1 = \sqrt{\frac{1-p_1}{2}}, \tag{50}$$

and

$$p_1 = \langle \alpha; m | - \alpha; m \rangle = \langle \alpha | (a^-)^m (a^+)^m | - \alpha \rangle, \tag{51}$$

for the subsystem 2

$$|\alpha\rangle = a_2(|0\rangle_2 + b_2|1\rangle_1; \qquad | - \alpha; m\rangle = a_2(|0\rangle_2 - b_2|1\rangle_2, \tag{52}$$

with

$$a_2 = \sqrt{\frac{1+p_2}{2}}; \qquad b_2 = \sqrt{\frac{1-p_2}{2}}, \tag{53}$$

and

$$p_2 = \langle \alpha | - \alpha \rangle. \tag{54}$$

Substituting the eqs.(52) and (49) into (5), we obtain the density matrix

$$\rho_{12} = \mathcal{N}_m^2 \begin{pmatrix} 2a_1^2a_2^2(1+p\cos k\pi) & 0 & 0 & 2a_1a_2b_1b_2(1-p\cos k\pi)) \\ 0 & 2a_1^2b_1^2(1-p\cos k\pi) & 2a_1a_2b_1b_2(1-p\cos k\pi)) & 0 \\ 0 & 2a_1a_2b_1b_2(1-p\cos k\pi) & 2a_2^2b_1^1(1-p\cos k\pi)) & 0 \\ 2a_1a_2b_1b_2(1-p\cos k\pi) & 0 & 0 & 2b_1^2b_2^2(1+p\cos k\pi) \end{pmatrix}, \tag{55}$$

with

$$\mathcal{N}_m = \sqrt{\frac{m! L_{m(-|\alpha|^2)}}{2m!(L_m(-|\alpha|^2 + p^3\cos(k\pi)L_m(|\alpha|^2)))}}. \tag{56}$$

For the matrix ρ_{23}

$$|\psi_{23}\rangle = \frac{\mathcal{N}^2}{\mathcal{N}_{23}^2}(|\alpha\rangle|\alpha\rangle + e^{im\pi}| - \alpha\rangle| - \alpha\rangle), \tag{57}$$

in this case we have

$$p = p_1 = p_2 = \langle \alpha | - \alpha \rangle, \tag{58}$$

with

$$a = a_1 = a_2 = \sqrt{\frac{1+p}{2}}; \qquad b = b_1 = b_2 = \sqrt{\frac{1-p}{2}}, \tag{59}$$

for ρ_{23}, we obtain the density matrix

$$\rho_{23} = \mathcal{N}_m^2 \begin{pmatrix} 2a^4(1+q_{23}\cos k\pi) & 0 & 0 & 2a^2b^2(1-q_{23}\cos k\pi)) \\ 0 & 2a^2b^2(1-q_{23}\cos k\pi) & 2a^2b^2(1-q_{23}\cos k\pi)) & 0 \\ 0 & 2a^2b^2(1-q_{23}\cos k\pi) & 2a^2b^2(1-q_{23}\cos k\pi)) & 0 \\ 2a^2b^2(1-q_{23}\cos k\pi) & 0 & 0 & 2b^4(1+q_{23}\cos k\pi) \end{pmatrix}, \tag{60}$$

where

$$q_{23} = p \times \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{61}$$

4 Quantifying the quantum discord

It has recently been shown that quantum discord can only be calculated for the limiting case of a quantum state has two qubits, but for analytical expression of a more general quantum state are not known until now. We have shown in this subsection the explicit form of quantum discord applied on the reduced density matricx described in (55) and (60) respectively, for which the discord can be evaluated in a simple way.

4.1 Generalities

The total correlation is usually quantified by the mutual information, usually expressed in term of von Neumann entropy as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{62}$$

where ρ_{AB} is the state of a bipartite quantum system composed of the subsystems A and B, the operator $\rho_{A(B)} = \mathrm{Tr}_{B(A)}(\rho_{AB})$ is the reduced state of $A(B)$ and $S(\rho)$ is the von Neumann entropy of a quantum state ρ.

Clearly, the mutual information $I(\rho_{AB})$ contains both quantum and classical correlations. Therefore, it is natural to decompose total correlations as

$$I(\rho_{AB}) = D(\rho_{AB}) + C(\rho_{AB}).$$

Let $\mathcal{B} = B_i$ be a POVM, then B_i should be a positive valued operator. Given the bipartite state ρ_{AB} a possible measure of the classical correlation between subsystems A and B is

$$C_B(\rho_{AB}) = max_{\mathcal{B}}[S(\rho_A) - \sum_i p_i S(\rho_A^i)], \tag{63}$$

$\mathcal{B}$ is a $POVM$ and the sum over i runs over all its elements B_i . The conditional density matrix ρ_A^i is the density matrix of A after performing the measurement B_i on B:

$$\rho_A^i = \frac{(\mathcal{B}^i \otimes II)\rho_{AB}(\mathcal{B}^i \otimes II)}{p_i}, \tag{64}$$

where $p_i = tr(\mathcal{B}^i \otimes II)\rho_{AB}(\mathcal{B}^i \otimes II)$ is the probability of A being in the state ρ_A^i.

It follows that, for a bipartite quantum system, the quantum discord is defined as the difference between total correlation and classical correlation([47], [48], [49]).

$$D(\rho_{AB}) = I(\rho_{AB}) - max_{\mathcal{B}}[S(\rho_A) - \sum_i p_i S(\rho_A^i)]. \tag{65}$$

4.2 Quantum discord

For the $X-$states ρ_{AB} (rank 2), which are writing in the form

$$\rho_{AB} = \begin{pmatrix} \rho_{00} & 0 & 0 & \sqrt{\rho_{00}\rho_{33}} \\ 0 & \rho_{11} & \sqrt{\rho_{11}\rho_{22}}) & 0 \\ 0 & \sqrt{\rho_{11}\rho_{22}} & \rho_{22} & 0 \\ \sqrt{\rho_{00}\rho_{33}} & 0 & 0 & \rho_{33} \end{pmatrix}. \tag{66}$$

the quantum discord $D_{AB} = D(\rho_{AB})$ is given by the main equation

$$D_{AB} = H(\rho_{00} + \rho_{11}) - H(\rho_{00} + \rho_{33}) + H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4(\sqrt{\rho_{00}\rho_{11}} - \sqrt{\rho_{22}\rho_{33}})^2}), \tag{67}$$

calculating the quantum discord in the states ρ_{12} as well as the states ρ_{13}, we resulting

$$\begin{aligned} D_{12} &= H(\mathcal{N}_m^2(1+p\cos k\pi)(1+pp_1)) && (68)\\ &- H(\mathcal{N}_m^2(1+p_1)(1+p^2\cos k\pi)) && (69)\\ &+ H(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\mathcal{N}_m^4(1-p^2)^2p_1^2}). && (70)\end{aligned}$$

Clearly, $D_{13}=D_{12}$. Then, the explicit expression of quantum discord $D(\rho_{23})$in the states ρ_{23} is given by the expression

$$\begin{aligned} D_{23} &= H(\mathcal{N}_m^2(1+p)(1+pq_{23}\cos k\pi)) && (71)\\ &- H(\mathcal{N}_m^2(1+q_{23}\cos k\pi)(1+p^2)) && (72)\\ &+ H(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\mathcal{N}_m^4(1-p^2)(1-q_{23}^2)}). && (73)\end{aligned}$$

4.3 Special case

- Reduced density matric ρ_{12}

We start with the special case $m=0$ (see figure 3), the state ρ_{12} is coincides with three-mode coherent states. At this point, we have $p_1=p$ and $L_0(\mid\alpha^2\mid)=1$, then the explicit expression of quantum discord for the density ρ_{12} is

$$\begin{aligned} D_{12} &= H(\frac{1}{2}\frac{(1+p\cos k\pi)(1+p^2)}{(1+p^3\cos k\pi)}) && (74)\\ &- H(\frac{1}{2}\frac{(1+p)(1+p^2\cos k\pi)}{1+p^3\cos k\pi)}) && (75)\\ &+ H(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{(1-p^2)^2p^2}{(1+p^3\cos k\pi)^2}}), && (76)\end{aligned}$$

in term of the overlap p. This expression of discord is equivalent with the discord given by [46] for n = 3, and the discord for symmetric (antisymmetric) states m even (m odd) as shown in the figure 1. Gives a plot of quantum discord versus the overlap p for the mixed state ρ_{12}. It must be noticed that for $k=0$, we find the maximum of discord obtained for the value p equal to 1/2. However, for $k=1$, the quantum discord takes the maximum value for $p\to 1$ and the discord increase with p increase.
The figure 4 gives a plot of quantum discord versus the number of photon for symmetric added entangled coherent states (i.e.k even) and for different value of m. As seen from the figure, after an initial increasing, the quantum discord decreases to vanish when the amplitude of photon $\mid\alpha^2\mid\to 2$. The minimum value of discord is obtained when has no photon excitation ($i.e.m=0$) and the maximum of quantum discord is obtained for m=10, we also found that the discord is depends on the number of the excitation photon. However, this maximum increases as m increases. In figure 5, we give a plot of the quantum discord for the antisymmetric case and for different excitation of photon number. It is remarkable that for antisymmetric quantum states the maximal value of quantum discord increases as m decreased contrarily to the symmetric states.

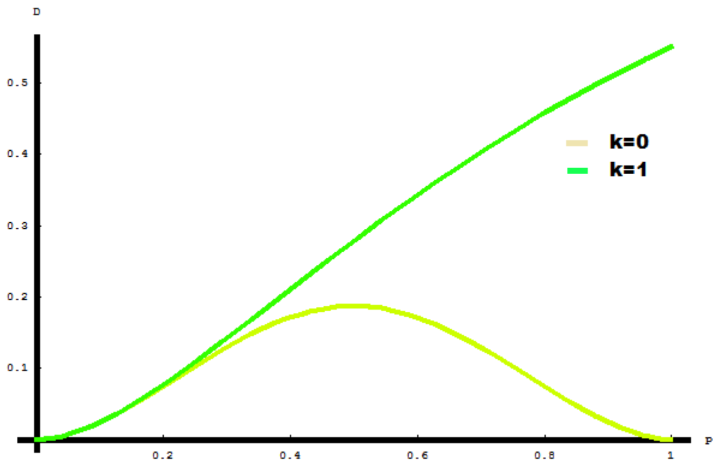

Figure 3: The quantum discord of SMEECS $|\psi(|\alpha|, m)\rangle$ versus $|\alpha|^2$ for (even) and (odd) states.

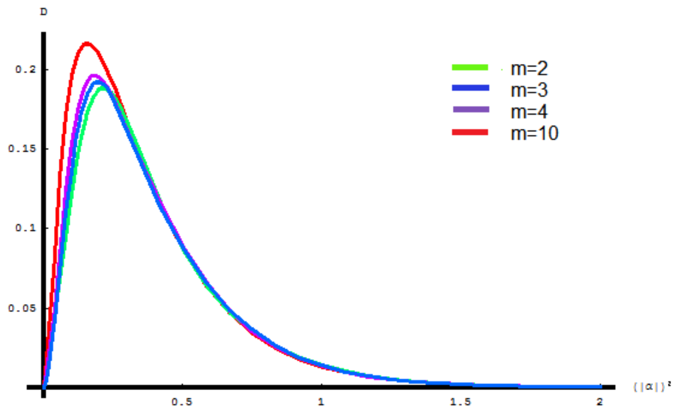

Figure 4: The quantum discord (even) of reduced density matric ρ_{12} versus $|\alpha|^2$ for the different number photon excitations.

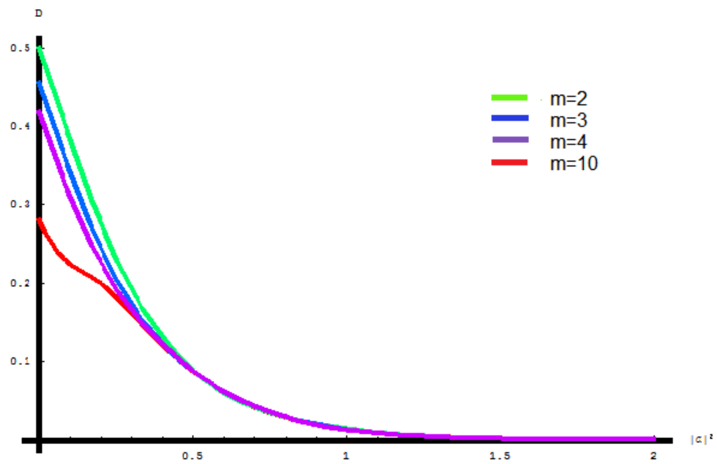

Figure 5: The quantum discord (odd) of reduced density matric ρ_{12} versus $|\alpha|^2$ for the different number photon excitations.

- Reduced density matric ρ_{23}

The explicit expression of quantum discord of reduced matric ρ_{23} in the special case $m = 0$ take the same result obtained for ρ_{12}.
The behavior of quantum discord of the reduced matric ρ_{23} versus $|\alpha|^2$ for symmetric (antisymmetric case)respectively is given in the figure 6 and 7 respectively. As seen from this figures, after an initial increasing, the quantum discord decreases to vanish when $|\alpha|^2 \longrightarrow 2$. It is clearly seen that the quantum discord increases with the photon excitation number is increases in the symmetric case. Otherwise, for the antisymmetric states, the quantum discord is decreases when the photon excitation number increases.
We observed that, the minimum value ($m = 2$) of quantum discord for the reduced matric ρ_{12} is higher than the maximum value ($m = 10$) obtained for the reduced matric ρ_{23}.

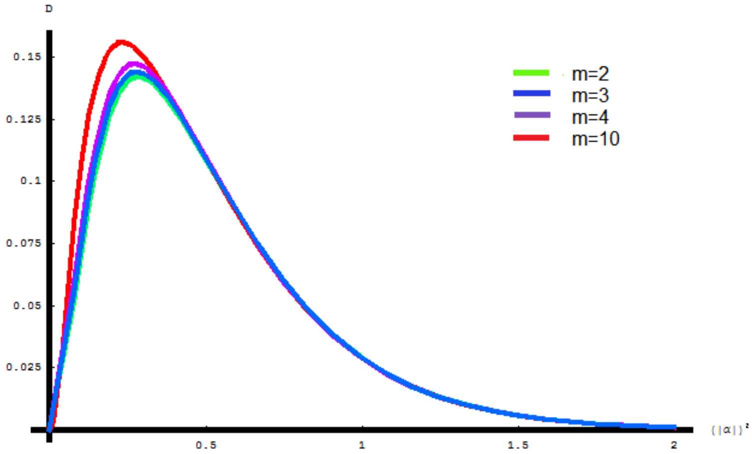

Figure 6: The quantum discord (odd) of reduced density matric ρ_{23} versus $|\alpha|^2$ for the different number photon excitations.

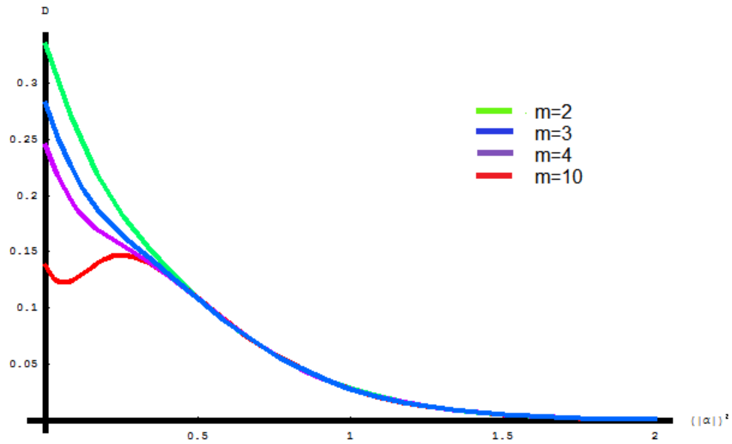

Figure 7: The quantum discord (odd) reduced density matric ρ_{23} versus $|\alpha|^2$ for the different number photon excitations.

5 Geometric measure of quantum discord

The main step in evaluating the quantum discord is the minimization of conditional entropy to get an explicit expression of the quantum discord in tripartite system. this process constitutes a hard task, even for two-qubit systems in a general state. This motivated the introduction of an alternative measure, which was named geometric quantum discord. The geometric measure of quantum discord has been recently proposed as a simple and intuitive quantifier of general non-classical correlations in bipartite quantum states, which can be exactly reformulated as the minimal disturbance, measured in terms of the squared Hilbert-Schmidt distance between the given quantum state ρ and the classical-quantum state χ, reading

$$D_g = min_{\chi}\|\rho - \chi\|^2, \tag{77}$$

where χ denotes the set of zero-discord states and $\|\rho-\chi\|^2 = Tr(\rho-\chi)^2$ is the Hilbert?Schmidt norm. Here, the measurement is performed on the subsystem A, the structure of the zero-discord state χ is defined by

$$\chi = \Sigma_{i=1,2} p_i|\psi_i\rangle\langle\psi_i| \otimes \rho_i, \tag{78}$$

ρ_i is the density operator and p_i is a probability distribution. Also, $\{|\psi_1\rangle\langle\psi_2|\}$ is an arbitrary orthonormed vector set.

For two-qubit systems, a general state can be written with Fano Bloch decomposition

$$\rho^{AB} = \frac{1}{4}[1 \otimes 1 + \Sigma_i^3(x_i\sigma_i^A \otimes 1 + y_i 1 \otimes 1 \otimes \sigma_i^B) + \Sigma_{i,j}R_{ij}\sigma_i^A \otimes \sigma_j^B], \tag{79}$$

with $x = x_i$, $y = y_i$ and R_{ij} the real parameters and $\sigma_{i=1,2,3}$ the Pauli matrices. We introduce a correlation matrix define by

$$\mathcal{R} = \begin{bmatrix} 1 & y^T \\ x & R \end{bmatrix}, \tag{80}$$

with $x = x_i$, $y = y_i$ (i=1,2,3) are the components in three dimension of local Bloch vectors and R a 3×3 matrix with the matrix elements $\mathcal{R}_{ij}$ form tak the following form

$$\mathcal{R}_{ij} = Tr(\rho\sigma_i^A \otimes \sigma_j^B) \qquad for \qquad i,j = 0,1,2,3. \tag{81}$$

Therefor, the concrete form of the geometric measure of quantum discord is defined by

$$D_g(\rho) = \frac{1}{4}(\|x\|^2 + \|R\|^2 - k_{max}). \tag{82}$$

Where K_{max} is the largest eigenvalue of the matrix k which can be written as $k = xx^T + RR^T$, and the quantity given by $\|x\|^2 + \|R\|^2$ present the summation of the eigenvalues of k.

So, we conclude that the GMQD can be analytically computed as following

$$D_g(\rho) = \frac{1}{4}min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \tag{83}$$

Noting that $\{\lambda_1, \lambda_2, \lambda_3\}$ present the eigenvalues of the matrix K, and the comparison of these eigenvalues is a key factor in the calculation of minimization procedure for GMQD. Specially, we concentrate on the bipartite mixed density mentioned in () and (55), these kinds of states belong to the so-called X-states whose algebraic characterization is presented in [50]. Thus, it can be written in the Bloch representation as follows

$$\rho_{ij} = \sum_{\alpha\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\alpha \tag{84}$$

.

- Reduced density matrix ρ_{12}

The correlation matrix elements $R_{\alpha,\beta}$ (α, β=0, 1, 2, 3) are given by
$R_{00} = 1$, $R_{11} = 2\mathcal{N}_m^2\sqrt{(1-p_m^2)((1-p^2))}$, $R_{22} = -2\mathcal{N}_m^2\sqrt{(1-p_m^2)((1-p^2))}p\cos k\pi$, $R_{33} = 2\mathcal{N}_m^2 p(p_m + \cos k\pi)$, $R_{03} = 2\mathcal{N}_m^2 p(1 + p_m\cos k\pi)$, $R_{30} = 2\mathcal{N}_m^2(p_m + p^2\cos k\pi)$.
With

$$\begin{aligned}\mathcal{N}_m &= \sqrt{\frac{m!L_{m(-|\alpha|^2)}}{2m!(L_m(-|\alpha|^2+p^3\cos(k\pi)L_m(|\alpha|^2)))}},\\ p_m &= p \times \frac{L_{m(-|\alpha|^2)}}{L_{m(-|\alpha|^2)}}.\end{aligned} \tag{85}$$

Having mapped the bipartite system ($\rho_{12} = \rho_{13}$) into a pair of two qubit presented in the previous subsection, we obtained the final expression of MGQD from a special comparison of the eigenvalues given by the matrix K, which are defined by

$$\lambda_1 = 4\mathcal{N}_m^4[(1+p^2)(p_m^2 + p^2) + 4p^2p_m\cos k\pi], \tag{86}$$

$$\lambda_2 = 4\mathcal{N}_m^4(1-p_m^2)(1-p^2), \tag{87}$$

$$\lambda_3 = 4\mathcal{N}_m^4(1-p_m^2)(1-p^2)p^2. \tag{88}$$

In the case, we obtained always $\lambda_2 \geq \lambda_3$,n and the maximum of the eigenvalues is λ_1 or λ_2. Thus, the equation (83) is reduced to a simpler expression

$$D_g = \frac{1}{4}min\{\lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \tag{89}$$

So, we need only to compare the two eigenvalues λ_1 and λ_2, for this we have two explicit expression of geometric quantum discord

$$D_g = \mathcal{N}_m^2 \quad [2p^2 + p_m^2(1+p^4) + p^2p_m\cos k\pi], \tag{90}$$

when the condition $\lambda_1 < \lambda_2$ is satisfied.
Otherwise, the form of quantum discord is given by

$$D_g = \mathcal{N}_m^2 \quad [(1-p_m^2)(1-p^4)]. \tag{91}$$

We will discuss the result obtained in equations.(90) and (91) in special cases m=0 and k=0, 1 respectively. Noting that for symmetric states (i.e k even) and m = 0, the condition which satisfied ($\lambda_1 < \lambda_2$) is $0 \leq p \leq \sqrt{2} - 1$, and the form of geometric measure of quantum discord described in equation (90) become

$$D_g = \frac{1}{4} \quad \frac{p^2(1+p^2)(2+(1-p^2)^2)}{(1+p^3)^2}. \tag{92}$$

In other hand, the condition obtained ($\sqrt{2}-1 \leq p \leq 1$) in the situation ($\lambda_2 < \lambda_1$) ensured that the MGQD stated in equation (91) can be rewritten in a simple structure.

$$D_g = \frac{1}{4} \quad \frac{p^2(1+p^2)(1-p^2)^2}{(1+p^3)^2}. \tag{93}$$

Therefore, for antisymmetric states (i.e. k odd) the result produced ($0 < p < 1$) when ($\lambda_1 < \lambda_2$) provided to define the MGQD as the following form

$$D_g = \frac{1}{4} \quad \frac{p^2(2+(1+p)^2)(1-p)^2}{(1+p^3)}. \tag{94}$$

The behavior of geometric quantum discord for special case is given in the figure 8. This figure gives a plot the geometric quantum discord versus the overlapping p for symmetric and antisymmetric SMEECSs. As seen from the figure (k=0), after an initial increasing, the quantum discord decreases to vanish when p = 1. The maximum value of quantum discord occurs when λ_1 and λ_2 coincide. It is remarkable that for antisymmetric quantum states take the maximum value when p = 1. The

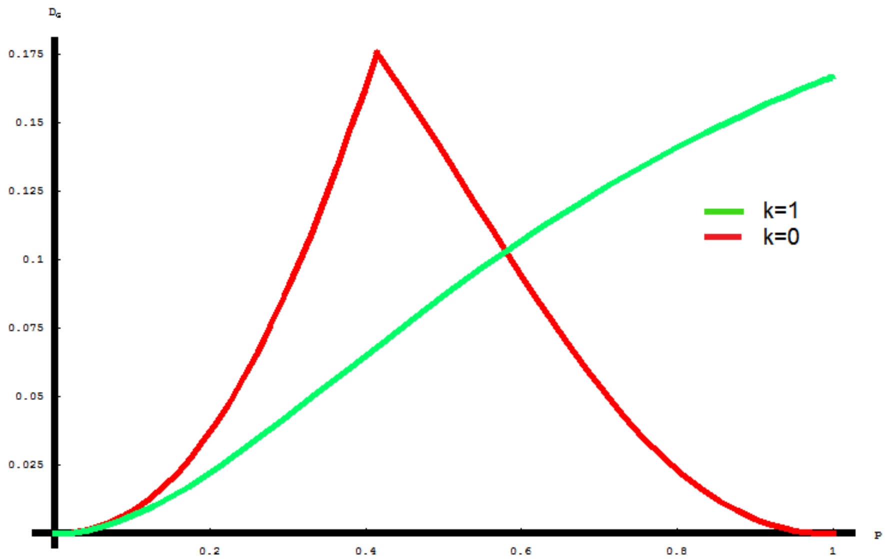

Figure 8: The GMQD for reduced density matric (ρ_{12}) of SMEECS versus p for m=0.

geometric measure of quantum discord for symmetric and antisymmetric states versus $|\alpha|^2$ are plotted. From the figure 9 symmetric SMEECSs (k even). We observe that when $|\alpha|^2 \to 1$ the quantum discord decreases to vanish. Thus, in the middle region λ_1 coincide with λ_2. Therefore, the quantum discord is maximal. We also observe that this maximum increases as the photon excitation number m increases. We can also observe through the figure 10, in the case m=2 the geometric quantum discord decreases rapidly from a maximum value to vanish if $|\alpha|^2 \to 1.3$. However, for the case $m \geq 3$, the quantum discord starts increasing to reach its maximal value ($i.e.\lambda_1 = \lambda_2$) and diminished when $|\alpha|^2 = 1, 3$. The result obtained is that, for antisymmetric quantum states containing more than particles, the maximal value of quantum discord increases as m increases contrarily to the symmetric states (see figure 9).

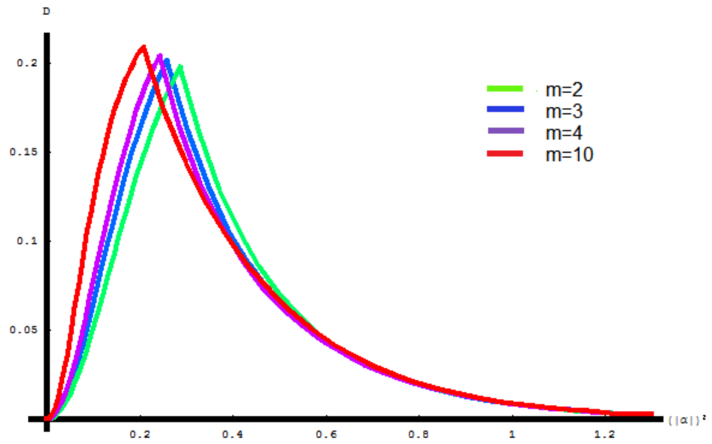

Figure 9: The GMQD for reduced density matric (ρ_{12}) of symmetric states of SMEECS versus $|\alpha|$ for the different number photon excitations.

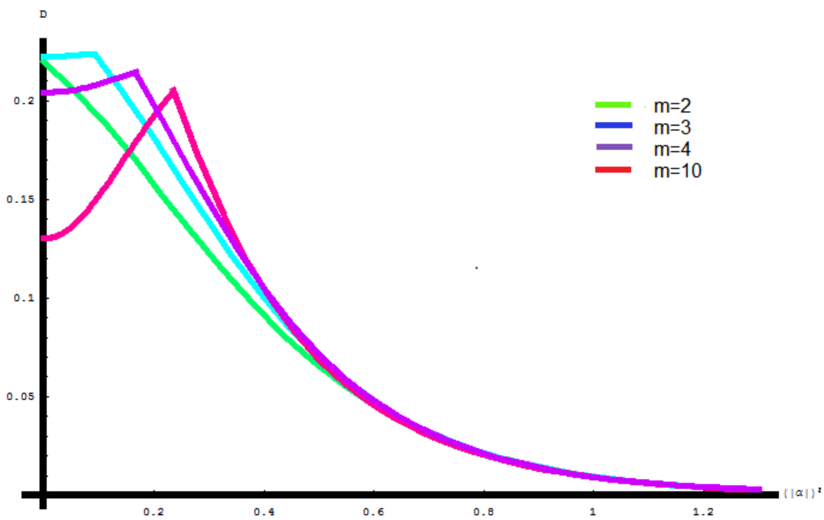

Figure 10: The GMQD for reduced density matric (ρ_{12}) of antisymmetric states of SMEECS versus $|\alpha|$ for the different number photon excitations.

- Reduced density matrix ρ_{23}

for calculated the eigenvalue of the reduced density matric ρ_{23}, we have follow the same procedure studied in reduced density matric.(ρ_{12}). Finaly we obtianed λ_1, λ_2 and λ_3 of the matrix ρ_{23} as

$$\lambda_1 = 4\mathcal{N}_m^4[(1+p^2)+(p^2+q_{23}^2)+4p^2p_m\cos k\pi], \quad (95)$$

$$\lambda_2 = 4\mathcal{N}_m^4(1-p^2)(1-p^2), \quad (96)$$

$$\lambda_3 = 4\mathcal{N}_m^4(1-p^2)(1-p^2)q_{23}^2. \quad (97)$$

The explicit expression of quantum discord of reduced matric ρ_{23} in the special case $m=0$ take the same result obtained for ρ_{12}.

The curve of geometric quantum discord of the reduced matric ρ_{23} versus $|\alpha|^2$ for symmetric and antisymmetric case is given in the figure 11, 12 respectively. As seen from this figures, after an initial increasing, the quantum discord decreases to vanish when $|\alpha|^2 \longrightarrow 1,3$. Thus, in the middle region λ_1 coincide with λ_2. Therefore, the quantum discord is maximal. It is clearly seen that the quantum discord decreases with the photon excitation number is increases in the symmetric case. Also, for the antisymmetric states, the quantum discord is decreases when the photon excitation number increases.

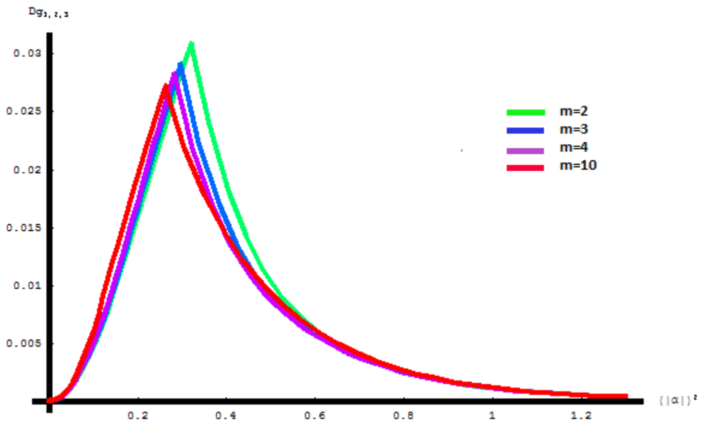

Figure 11: The GMQD for reduced density matric (ρ_{23})of symmetric states of SMEECS versus $|\alpha|^2$ for the different number photon excitations.

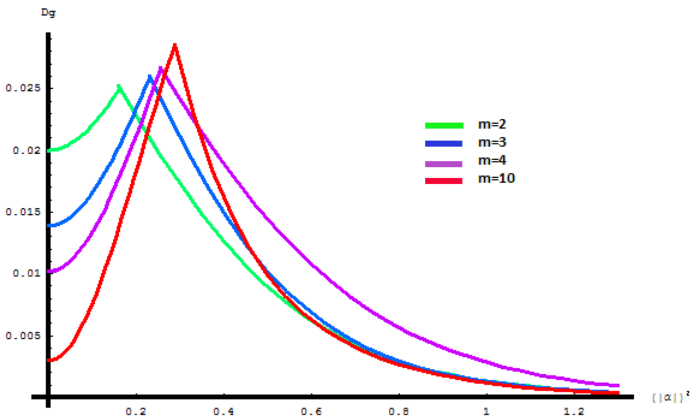

Figure 12: The GMQD for reduced density matric (ρ_{23}) of antisymmetric states of SMEECS versus $|\alpha|^2$ for the different number photon excitations.

6 Monogamy of quantum discord for a three-qubit entangled state

Monogamy of quantum correlations is a property satisfied by certain entanglement measures in a multipartite scenario. Given a tripartite state ρ_{123}, the monogamy condition for a bipartite quantum correlation measure $\mathcal{Q}$ assures that the bipartite quantum correlations in the density operator $\rho_{1|23}$ are distributed in such a way that the following inequality is satisfied

$$\mathcal{Q}(\rho_{1|23}) \geq \mathcal{Q}(\rho_{12}) + \mathcal{Q}(\rho_{13}). \quad (98)$$

Violation of the above inequality will imply that the quantity $\mathcal{Q}$ is polygamous for the corresponding state. Otherwise, this inequality is sufficient for quantum discord to be monogamous.
To illustrate the above analysis, we will investigate the properties of quantum discord monogamy in two different ways (quantum discord and geometric quantum discord using norm 2).

- Monogamy of quantum discord

To investigate the monogamy relation of quantum discord measured in quantum systems involving three qubits, Coffman et al [28] introduced the so called tripartite state equation (30). It is defined as

$$D(\rho_{1|23}) \geq D(\rho_{12}) + D(\rho_{13}), \tag{99}$$

where 1, 2 and 3 mean the respective parts of a tripartite system. Note that here $D_{1|23}$ is given by the entanglement between qubit (1) and the joint qubits (23). The quantum discord coincides with the entanglement of formation

$$D(\rho_{1|23}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2(\rho_{1|23})}) = E(\rho_{1|23}), \tag{100}$$

with

$$C(\rho_{1|23}) = (\frac{L_m(-|\alpha|^2)}{L_m(-|\alpha|^2) + \cos(k\pi)e^{-6|\alpha|^2}L_m(|\alpha|^2)}) \ \ (1 - (e^{-8|\alpha|^2})) \ \ (1 - (e^{-2|\alpha|^2}\frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)})^2). \tag{101}$$

We will now present the conditions that signal whether a tripartite quantum state is monogamous in nature with respect to quantum discord. In figure 13, the monogamy quantity D_{123} is plotted as functions of overlapping p, for symmetric state the inequality mentioned as the above is satisfied. Otherwise, for the antisymmetric state (i.e.k=1) the monogamy relation is violated during the interval $0 \leq p \leq 1$.
In the figures 14 and 15, corresponding respectively to symmetric and antisymmetric added coherent states, we plot the monogamy quantity as a function of $|\alpha|^2$. We examine the positivity of the following inequality

$$D_{123} = D_{123}(m, |\alpha|^2) = D(\rho_{1|23}) - D(\rho_{12}) - D(\rho_{13}), \tag{102}$$

defined in terms of the bipartite quantum discord. We shall restrict our discussion in what follows to the interval $0 \leq |\alpha|^2 \leq 1,3$. Clearly, for symmetric states (see figure 14) the function described by (102)is non positive when $0 \leq |\alpha|^2 \leq 0.644$, consequently the quantum discord is non monogamous. Otherwise, if $|\alpha|^2 \geq 0.644$ the quantum discord is monogamous whatever the photon excitations number m. Thus, for the antisymmetric states (figure 15) the plotted curve indicates that the inequality of monogamy (102) is not satisfied if $0 \leq |\alpha|^2 \leq 0.4$, Otherwise, for $|\alpha|^2 \geq 0.4$ the quantity D_{123} is positive and the quantum discord is monogamous for everything values m.

- Monogamy of geometric quantum discord

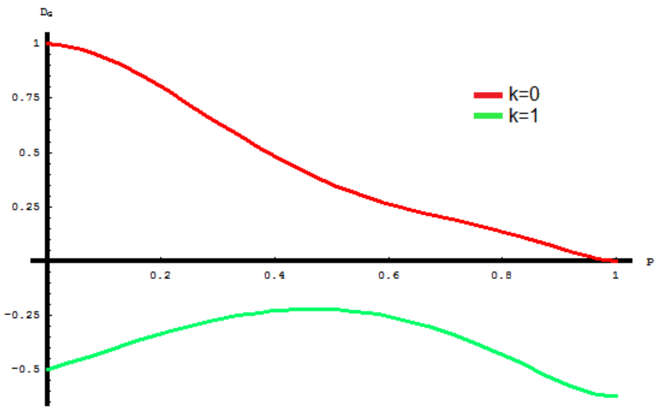

Figure 13: Monogamy D_{123} versus the overlapping p when m=0

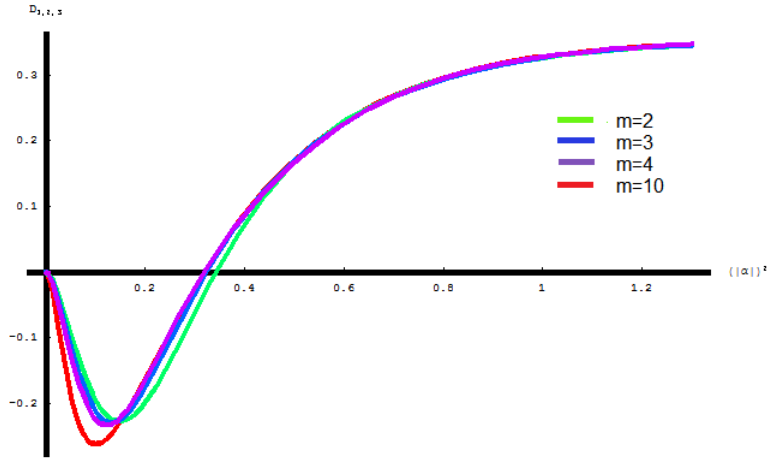

Figure 14: Monogamy D_{123} of symmetric states versus $|\alpha|^2$ for different number photon excitations.

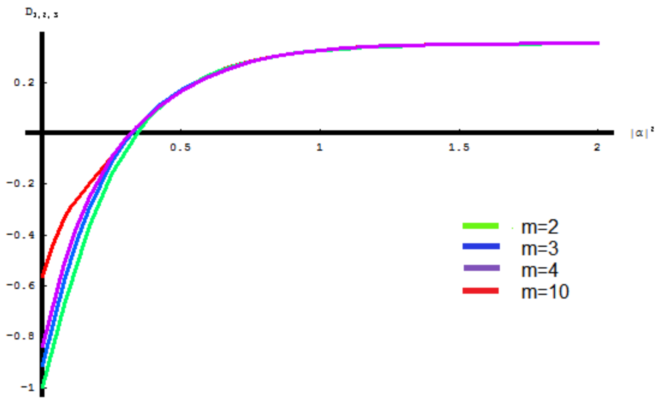

Figure 15: Monogamy D_{123} of antisymmetric states versus $|\alpha|^2$ for different number photon excitations.

In this part, we will examine whether the geometric quantum discord of a tripartite state ρ_{123} wich satisfy the following inequality

$$\mathcal{D}_g(\rho_{1|23} \geq \mathcal{D}_g(\rho_{12}) + \mathcal{D}_g(\rho_{13}). \tag{103}$$

In the pure bi-partitioning scheme 123, it is easy to check, using the method presented in [27], that the geometric discord is related to the concurrence of the state $\rho(1|23)$ as follows

$$\mathcal{D}_g(\rho_{1|23}) = \frac{1}{2}\mathcal{C}^2_{1|23}. \tag{104}$$

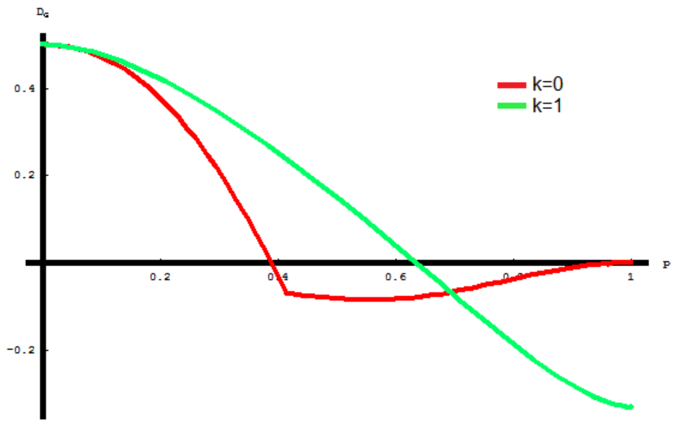

Figure 16: Monogamy $D_g(123)$ of geometric quantum discord as a function of overlapping for .

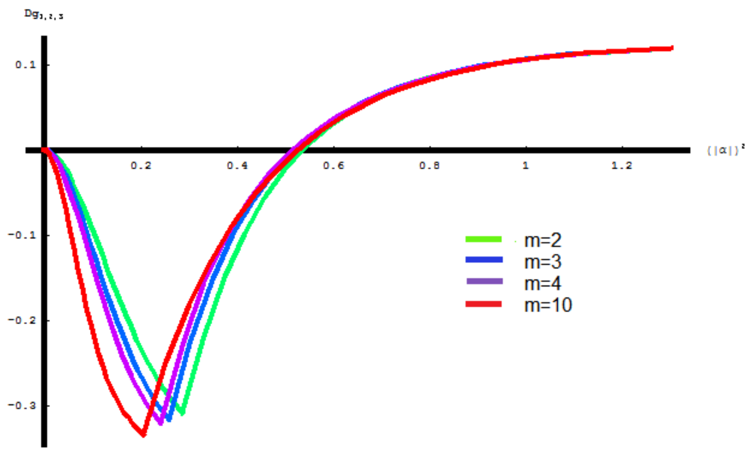

Figure 17: Monogamy of geometric quantum discord versus $|\alpha|^2$ for symmetric states and for different values of m.

Using the result obtained in the previous subsection. Thus, We treat first the Monogamy of the geometric quantum discord for symmetric states (k = 0, 1 respectively) and for the special case(m = 0), we have

$$D_g(\rho_{12}) = D_g(\rho_{13}) = \frac{3p^2 - 4p^3 + p^6}{4(1+p^3)}, \tag{105}$$

for $0 \leq p \leq \sqrt{2} - 1$ and

$$D_g(\rho_{12}) = D_g(\rho_{13}) = \frac{(1-p^2)(1-p^4)}{4(1+p^3)}, \tag{106}$$

if the condition $\sqrt{2} - 1 \leq p \leq 1$ is satisfied. we have also

$$D_g(\rho_{1|23}) = \frac{1}{2}\frac{(1-p^4)^{?2}(1-p_m^2)^2}{(1+p^3)^2}, \tag{107}$$

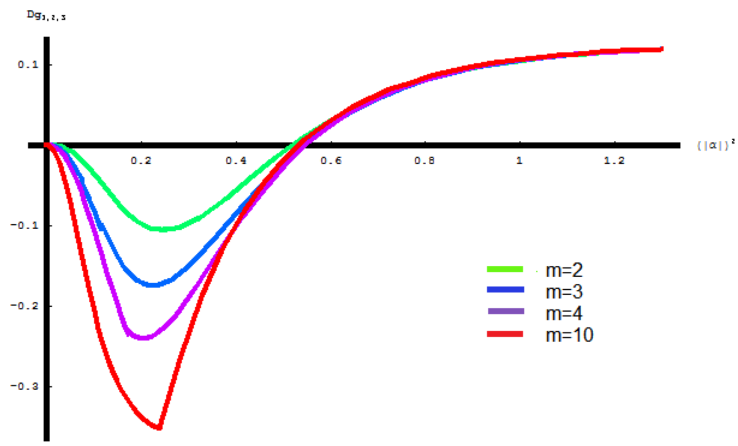

Figure 18: Monogamy of geometric quantum discord versus $|\alpha|^2$ for antisymmetric states and for different values of m.

for $0 \leq p \leq 1$. So, we can define the quantity $D_g(123)$ as the following form

$$D_g(123) = D_g(\rho_{1|23}) - D_g(\rho_{12}) - D_g(\rho_{13}). \tag{108}$$

The quantity indicate whether above is plotted in figure 16 versus the overlap p. Clearly, the geometric quantum discord is monogamous for the tripartite states with p such that $0 \leq p \leq 0.388$. Contrariwise, the geometric quantum discord is discord polygamous.

For antisymmetric states(k = 1), the quantum discord take the following form

$$D_g(\rho_{13}) = D_g(\rho_{12}) = \frac{1}{4} \ [\frac{3p^2 - 4p^3 + p^6}{4(1-p^3)^2}], \tag{109}$$

when the condition $0 \leq p \leq 1$ is satisfied. And for $D_g(\rho 1/23)$, we have

$$D_g(\rho_{1|23}) = \frac{1}{2}[\frac{(1-p^2)^2(1-p^4)^2}{4(1-p^3)^2}], \tag{110}$$

for $0 \leq p \leq 1$. Therefor, the numerical results reported in figure 16 show that the geometric discord is monogamous when p such that $0 \leq |\alpha|^2 \leq 0.61649$. Otherwise, the geometric quantum discord is polygamous.

Thereafter, we plot the quantity $D_g(123)$ for the different values m and investigate the influence of photon excitations on monogamy of geometric discord. So, for symmetric case (figure 17) the behaviour of quantum discord the monogamy inequality is not satisfied for $0 \leq |\alpha|^2 \leq 0.524$. In the case where $|\alpha|^2 > 0.524$ the monogamy quantity is monotonically increasing for everything values m and the geometric quantum discord is monogamous. In the other hand, for antisymmetric states(i.e.k=1) with different value m (see figure 18), the monogamy is satisfied when $|\alpha|^2 \geq 0.53$, and this quantity becomes violated for everything values m if $0 \leq |\alpha|^2 \leq 0.53$.

7 Concluding remarks

Summarizing, we have presented in the early of this paper a class of the single mode excited entangled coherent states (SMEECSs)$|\psi_p(\alpha, m)\rangle$, which are obtained through actions of creation operator on the entangled coherent states. Then, we have exhibited the important properties of quantum entanglement by using different ways (specially, the concurrence, quantum discord and its version geometric). The first way, we have studied the concurrence for bipartite systems and investigated the influence of phonon excitations numbers on quantum entanglement. We also employed the other process for studied the quantum correlation of add coherent states for tripartite quantum states (see the equations (60) and (55) by the quantum discord. Thus, we found two explicit analytic expressions of this measure and the results obtained are discussed. Another way which treated the quantum correlations by introducing the geometric version of quantum discord, at this stage we derived a necessary and sufficient condition. Specially, for the case of three-qubit states we have proposed in the our discussion, two version(i.e.symmetric, antisymmetric states, respectively) and the result obtained is explained in terms of different number photon excitations(i.e.the influence of m on geometric quantum discord). To close our work, we have employed the concept of quantum monogamy corresponding to quantum discord and its geometric version. In particular, we have investigated the relation between discord monogamy and a genuine tripartite entanglement measure for three-qubit pure states. Therefor, We have demonstrated that the quantum correlations examined by the entropic measure, geometric measure respectively does not satisfy the monogamy relation(98). A very important result is derived in this work, from a value determined of $|\alpha|^2$, we see that no effect of the addition of the photon can be found on the measurement of monogamy. The analysis presented in this letter can be extended to the effect of subtracting the photon of tripartite GHZ coherent states.

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