

# A polynomial class of $u(2)$ algebras

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## Abstract

A  $r$ -parameter  $u_{\{\kappa_1, \kappa_2, \dots, \kappa_r\}}(2)$  algebra is introduced. Finite unitary representations are investigated. This polynomial algebra reduces via a contraction procedure to the generalized Weyl-Heisenberg algebra  $\mathcal{A}_{\{\kappa_1, \kappa_2, \dots, \kappa_r\}}$  (M. Daoud and M. Kibler, J. Phys. A: Math. Theor. **45** (2012) 244036). A pair of nonlinear (quadratic) bosons of type  $\mathcal{A}_\kappa \equiv \mathcal{A}_{\{\kappa_1=\kappa, \kappa_2=0, \dots, \kappa_r=0\}}$  are used to construct, à la Schwinger, a one parameter family of (cubic)  $u_\kappa(2)$  algebra. The corresponding Hilbert space is constructed. The analytical Bargmann representation is also presented.

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# 1 Introduction

Nonlinear extensions of finite Lie algebras continue to attract much attention both in mathematics and physics. The special interest in nonlinear algebras is mainly motivated by their appearance as dynamical symmetries for several physical systems (two identical particles in two dimensional manifold, isotropic oscillator and Kepler system in a two dimensional curved space, ...) They have also found various applications in other areas of physics such as quantum statistics, quantum optics, etc. [1-17]. For bosonic realizations of such algebras, different kinds of generalized boson algebra were introduced in the literature. Different extensions give rise to different kinds of structure relations and subsequently different types of unitary representations. One may quote for instance the concept deformed boson ( $q$ -boson) algebra, firstly proposed by Arik and Coon [18] and developed further by Macfarlane [19] and Biedenharn [20], which was used extensively in the literature to provide bosonic realizations of quantum algebras.

Now, it is well established that all generalized or deformed bosons can be accommodated within a unified framework in which the product of creation and annihilation operators is a function of the number operator (see for instance [21]). This structure function reflects the nonlinearity effects and the deviation from the usual boson algebra. Of particular interest are structure functions which expand as polynomials with respect in the number operator. This special kind of nonlinear oscillator algebra is commonly called in the literature polynomial Weyl-Heisenberg algebras (see [22, 23, 24] and references quoted therein). In this context, recently, the  $r$ -parameter Weyl-Heisenberg algebra  $\mathcal{A}_{\kappa_1, \kappa_2, \dots, \kappa_r}$  was introduced. It generalizes the single mode algebra  $\mathcal{A}_\kappa$  which covers the usual boson,  $su(2)$ ,  $su(1, 1)$  algebras [22]. Also, this algebra turns out to be of special interest in dealing with some one dimensional quantum potentials (for more detail see [22]). The representation theory of algebras of type  $\mathcal{A}_{\kappa_1, \kappa_2, \dots, \kappa_r}$  is very rich. Indeed, it has been shown that they admit finite and finite dimensional unitary representations depending on the values of the parameters  $\kappa_1, \kappa_2, \dots, \kappa_r$ . Multi-mode extensions of the algebra  $\mathcal{A}_\kappa$  and the related representations were investigated in [23, 25]. Furthermore, the fermionic analogue of the  $\mathcal{A}_\kappa$  algebra was proposed in [26, 27].

In addition, the generalized  $su(2)$  algebras developed in the context of the theory of quantum algebras have introduced in a way that their finite dimensional unitary representations remain as close as possible to usual one. The interest in such nonlinear algebraic structure is due to their relevance in diverse areas such as integrable models [28, 29, 30, 31, 32] and super-symmetric quantum systems [32, 33]. Similarly, to generalized boson algebras, the nonlinear  $u(2)$  algebras can be defined within an unified framework in which the commutator, involving the raising and lowering generators  $J_+$  and  $J_-$ , is expanded as formal power series of the Cartan subalgebra  $J_0$  and  $J_3$ . Unitary irreducible representations were investigated for some specific forms of the polynomial structure function  $f(J_3) \equiv [J_+, J_-]$  (see for instance [14]).

In this work, we shall focus on the polynomial extension of  $u(2)$  algebra by adopting a procedure similar to one giving the algebra  $\mathcal{A}_{\kappa_1, \kappa_2, \dots, \kappa_r}$  [23]. In this scenario, we introduce the  $r$ -parameter  $u_{\kappa_1, \kappa_2, \dots, \kappa_r}(2)$ . The structure function defined by means of the commutator  $[J_+, J_-]$  is a polynomial of order  $r$  in the Cartan generators  $J_0$  and  $J_3$ . The second facet of this paper concerns the Schwinger realization of a polynomial  $u(2)$  algebra by means of a pair of bosons of type  $\mathcal{A}_\kappa$  [22].

This paper is organized as follows. Section 2 deals with a  $r$ -parameter extension of  $u(2)$  algebra. For this polynomial variant, denoted by  $u_{\kappa_1, \kappa_2, \dots, \kappa_r}(2)$ , we investigate the finite dimensional representations which clearly will depend on the values taking by the parameters  $\kappa_1, \kappa_2, \dots, \kappa_r$ . We show by a contrarction procedure that the algebra  $u_{\kappa_1, \kappa_2, \dots, \kappa_r}(2)$  reduces to the generalized Weyl-Heisenberg algebra  $\mathcal{A}_{\kappa_1, \kappa_2, \dots, \kappa_r}$  introduced in [24]. Section 3 is devoted to the Schwinger realization of another kind of polynomial  $u(2)$  algebras. The corresponding Fock space is explicitly constructed. Analytical representations and the associated set of coherent states are also derived. Concluding remarks close this paper.

## 2 Polynomial $u_{\{\kappa\}}(2)$ algebra

### 2.1 The algebra

A generalized  $u(2)$  algebra, defined on  $\mathbb{C}$ , is spanned by four linear operators  $J_\alpha$  ( $\alpha = 0, 3, +, -$ ) satisfying the commutation relations

$$[J_+, J_-] = G(J_3, J_0) \quad [J_3, J_\pm] = \pm J_\pm \quad [J_0, J_\alpha] = 0. \quad (1)$$

Since we are interested in unitary representations, we require the following hermitian conjugation conditions

$$J_+ = (J_-)^\dagger \quad J_0 = J_0^\dagger \quad J_3 = J_3^\dagger. \quad (2)$$

Thus, the  $G$  function in (1) should satisfy

$$G(J_3, J_0) = (G(J_3, J_0))^\dagger. \quad (3)$$

It must be emphasized that different polynomial extensions were considered in the literature leading to different structures relations and consequently different types of representations (a complete list of references can be found in [34]). In this work, we shall consider the  $r$ -parameters  $u_{\{\kappa\}}(2) \equiv u_{\{\kappa_1, \kappa_2, \dots, \kappa_r\}}(2)$  algebra characterized by the structure function  $\Phi(J_3, J_0)$  defined by

$$J_+ J_- = \Phi(J_3, J_0) \quad J_- J_+ = \Phi(J_3 + 1, J_0) \quad (4)$$

so that the the  $G$  function occurring in the commutator between the Weyl generators  $J_+$  and  $J_-$  (see eq.(1)) writes

$$G(J_3, J_0) = \Phi(J_3, J_0) - \Phi(J_3 + 1, J_0). \quad (5)$$

The function  $\Phi$ , specifying the nonlinear scheme in this work, is defined by

$$\Phi(J_3, J_0) = [J_0 + J_3][I + J_0 - J_3][I + \kappa_1(J_0 + J_3 - I)][I + \kappa_2(J_0 + J_3 - I)] \cdots [I + \kappa_r(J_0 + J_3 - I)] \quad (6)$$

where the  $\kappa_i$ 's ( $i = 1, 2, \dots, r$ ) are real parameters and  $I$  stands for the unit operator. The structure function (6) can be expanded as

$$\Phi(J_3, J_0) = [J_0 + J_3][I + J_0 - J_3] \sum_{i=0}^r s_i (J_0 + J_3 - I)^i. \quad (7)$$

In the last expression, the coefficients  $s_i$  are given by

$$s_0 = 1 \quad s_i = \sum_{j_1 < j_2 < \dots < j_i} \kappa_{j_1} \kappa_{j_2} \cdots \kappa_{j_i} \quad (i = 1, 2, \dots, r) \quad (8)$$

where the indices  $j_1, j_2, \dots, j_i$  take the values  $1, 2, \dots, r$ . Reporting (7) in (5), one verifies

$$G(J_3, J_0) = 2J_3 + \Delta_{\{\kappa\}} \quad (9)$$

where the finite difference operator  $\Delta_{\{\kappa\}}$  takes the form

$$\Delta_{\{\kappa\}} = [J_0 + J_3][I + J_0 - J_3] \sum_{i=0}^r s_i (J_0 + J_3 - I)^i - [J_0 + J_3 + I][J_0 - J_3] \sum_{i=0}^r s_i (J_0 + J_3)^i. \quad (10)$$

In the limiting case  $\{\kappa\} \equiv \{0, 0, \dots, 0\}$ , one verifies  $\Delta_{\{\kappa\}} = 0$  and hence the structure relations (1) reduces to usual ones and the ordinary  $u(2)$  algebra is recovered.

## 2.2 Hilbert space

Similar to standard  $u(2)$  case, the unitary representations can be determined by considering a basis  $\{|j, m\rangle, m = -j, -j + 1, \dots, j - 1, j\}$ , characterized by an integer or half integer  $j$ , in which both Cartan subalgebra generators  $J_0$  and  $J_3$  are diagonal

$$J_0|j, m\rangle = j|j, m\rangle \quad J_3|j, m\rangle = m|j, m\rangle. \quad (11)$$

In this basis, the structure function acts as

$$\Phi(J_3, J_0)|j, m\rangle = \Phi(j, m)|j, m\rangle. \quad (12)$$

with

$$\Phi(j, m) = [j + m][1 + j - m][1 + \kappa_1(j + m - 1)][1 + \kappa_2(j + m - 1)] \cdots [1 + \kappa_r(j + m - 1)]. \quad (13)$$

The action of raising and lowering generators write as follows

$$J_+|j, m\rangle = \sqrt{\Phi(j, m)}|j, m + 1\rangle \quad J_-|j, m\rangle = \sqrt{\Phi(j, m + 1)}|j, m - 1\rangle. \quad (14)$$

It is worth to note that the structure function is the product of the operators  $J_+$  and  $J_-$  (4) and therefore the eigenvalues  $\Phi(j, m)$  should be non negative. Thus, the following condition

$$[j + m][1 + j - m][1 + \kappa_1(j + m - 1)][1 + \kappa_2(j + m - 1)] \cdots [1 + \kappa_r(j + m - 1)] \geq 0 \quad (15)$$

must be fulfilled. Clearly, the positivity of the product in the left-hand side of this inequality depends on the parameters  $\kappa_1, \kappa_2, \dots, \kappa_r$ . This determines the dimension of the representations space of nonlinear  $u(2)$  algebras defined through the structure function (4). Obviously, when all the parameters  $\kappa_i$  ( $i = 1, 2, \dots, r$ ) are in  $\mathbb{R}_+$ , the condition (15) is satisfied. In this case, the algebra  $u_{\{\kappa\}}(2)$  admits a finite dimensional representations characterized by integer or half integer  $j$ . The dimension is exactly  $2j + 1$  ( $m = -j, -j + 1, \dots, j$ ) as for the ordinary  $u(2)$  algebra. The situation changes when one or more parameters are negative. Indeed, suppose that all parameters are positive except one, say  $\kappa_i$  ( $1 \leq i \leq r$ ). In this situation, the condition (15) gives

$$m = -j, -j + 1, \dots, -j + E\left(-\frac{1}{\kappa_i}\right) \quad (16)$$

where the symbol  $E(x)$  stands for the integer part of  $x$ . To simplify, we assume that  $-1/\kappa_i \in \mathbb{N}^*$ . Accordingly, the dimension of the Hilbert space is

$$d = \inf(2j + 1, d_i)$$

with

$$d_i = 1 - \frac{1}{\kappa_i}.$$

In this respect for  $\kappa_i < -\frac{1}{2j}$ , the  $(2j + 1)$ -dimensional representation space is truncated from  $(2j + 1)$  to  $d_i$ . This dimensional truncation can be traduced by the following nilpotent properties of the operators  $J_+$  and  $J_-$

$$(J_-)^{d_i} = (J_+)^{d_i} = 0$$

which differ from the usual case. Here, we have deliberately focused on the case where only one parameter is non positive. However, we stress that these analysis can be repeated similarly to determine the representation space dimension of  $u_{\{\kappa\}}(2)$  involving two, three or more negative parameters.

### 2.3 Contraction and $\mathcal{A}_{\{\kappa\}}$ algebra

In order to establish the correspondence between the polynomial algebra  $u_{\{\kappa\}}(2)$  and the generalized Weyl-Heisenberg  $\mathcal{A}_{\{\kappa\}}$  algebra introduced in [24], we review the Schwinger map in realizing the ordinary algebra  $u(2)$ . In this realization, the generators are defined by means of two commuting pairs of ordinary (un-deformed or linear) bosons, say  $\{b_+^+, b_+^-\}$  and  $\{b_-^+, b_-^-\}$  acting on a two-particle Fock space  $\mathcal{F} = \{|n_+, n_-\rangle : n_+ \in \mathbb{N}, n_- \in \mathbb{N}\}$ . The passage from boson state vectors  $|n_+, n_-\rangle$  to angular momentum state vectors  $|j, m\rangle$  is achieved via the relations

$$j := \frac{1}{2}(n_+ + n_-), \quad m := \frac{1}{2}(n_+ - n_-) \quad (17)$$

and

$$|j, m\rangle \equiv |j + m, j - m\rangle = |n_+, n_-\rangle. \quad (18)$$

In this picture, the actions of boson operators may thus be rewritten as

$$\begin{aligned} b_{\pm}^{\pm} |j, m\rangle &= \sqrt{j \pm m + \frac{1}{2} + \frac{1}{2}} |j + \frac{1}{2}, m \pm \frac{1}{2}\rangle, \\ b_{\pm}^{\mp} |j, m\rangle &= \sqrt{j \pm m + \frac{1}{2} - \frac{1}{2}} |j - \frac{1}{2}, m \mp \frac{1}{2}\rangle, \end{aligned} \quad (19)$$

so that the bosons behave as ladder operators for the quantum numbers  $j$  and  $m$  (with  $|m| \leq j$ ). In this realization, the four operators  $J_{\alpha}$  ( $\alpha = 0, 3, +, -$ ) are expressed as

$$J_0 := \frac{1}{2}(N_+ + N_-), \quad J_3 := \frac{1}{2}(N_+ - N_-), \quad J_+ := b_+^{\dagger} b_-, \quad J_- := b_-^{\dagger} b_+ \quad (20)$$

in terms of usual creation, annihilation and number operators ( $N_{\pm} = b_{\pm}^{\dagger} b_{\pm}$ ). They satisfy the commutation rules

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_3, \quad [J_0, J_{\alpha}] = 0. \quad (21)$$

By restricting the Fock space  $\mathcal{F}$  to its subspace generated by the vectors such that  $n_+ + n_- = 2j$ , one verifies the identities

$$J_0 + J_3 = N \quad J_0 - J_3 = 2jI - N, \quad (22)$$

where we replaced the number operator  $N_+$  by  $N$ . Consequently, the structure function (6) takes now the form

$$\Phi(J_3, J_0) \equiv \Phi_j(N) = N((2j+1)I - N)(I + \kappa_1(N - I))(I + \kappa_2(N - I)) \cdots (I + \kappa_r(N - I)). \quad (23)$$

Mimicking the contraction procedure to pass from  $su(2)$  algebra to harmonic oscillator algebra, we introduce the operators

$$a_+ = \frac{J_+}{\sqrt{2j}}, \quad a_- = \frac{J_-}{\sqrt{2j}}, \quad (24)$$

and considering the situation where  $j$  is large, one obtains

$$a_+ a_- = \Phi_{\infty}(N) = N(I + \kappa_1(N - I))(I + \kappa_2(N - I)) \cdots (I + \kappa_r(N - I)). \quad (25)$$

This is exactly the structure function  $F(N) \equiv \Phi_{\infty}(N)$  defining the generalized Weyl-Heisenberg  $\mathcal{A}_{\{\kappa\}}$  introduced in [24]. As a particular case, for  $r = 1$  and  $\kappa_1 = \kappa$ , the algebra  $\mathcal{A}_{\{\kappa\}}$  reduces to the generalized Weyl-Heisenberg algebra  $\mathcal{A}_{\kappa}$  [22]. The commutation relations reduce to

$$[a^-, a^+] = I + 2\kappa N \quad [N, a^{\pm}] = \pm a^{\pm}. \quad (26)$$

The corresponding structure function is now quadratic with respect to number operator [22]

$$F(N) \equiv a^+ a^- = N(I + \kappa(N - I)). \quad (27)$$

To close this section, it is interesting to note that Schwinger realizations of non linear  $su(2)$  algebras were studied for some specific nonlinear scheme in [35, 36]. In this sense, it is natural to ask: What kind of nonlinear  $u(2)$  algebra can be realized in the Schwinger sense in terms of a pair of two generalized bosons of type  $\mathcal{A}_{\kappa}$  (26)? This issue constitutes the main of the following section.

### 3 Two $\mathcal{A}_\kappa$ -boson realization of polynomial $u(2)$ algebras

#### 3.1 The realization

In this section, we extend the usual Schwinger realization by replacing ordinary bosons by objects satisfying the commutation rules of  $\mathcal{A}_\kappa$  algebra. In this order, we consider two pairs of mutually commuting generalized Weyl-Heisenberg algebra  $\mathcal{A}_\kappa$  (26). Each is spanned by three linear operators  $a_i^-$ ,  $a_i^+$  and  $N_i$  ( $i = 1, 2$ ) satisfying the relations

$$[a_i^-, a_i^+] = I + 2\kappa N_i \quad [N_i, a_i^\pm] = \pm a_i^\pm \quad (a_i^-)^\dagger = a_i^+ \quad N_i^\dagger = N_i. \quad (28)$$

We denote the corresponding Fock space, finite or infinite dimensional, by

$$\mathcal{F}_\kappa = \mathcal{F}_1 \otimes \mathcal{F}_2 = \{|n_1, n_2\rangle, n_1, n_2 \text{ ranging}\}. \quad (29)$$

The finiteness of  $\mathcal{F}_\kappa$  depends on the value of the parameter  $\kappa$ . The actions of creation, annihilation and number operators on  $\mathcal{F}_\kappa$  are defined by

$$\begin{aligned} a_i^+ |n_1, n_2\rangle &= \sqrt{F(n_i + 1)} |n_1 + s_i^-, n_2 + s_i^+\rangle, \\ a_i^- |n_1, n_2\rangle &= \sqrt{F(n_i)} |n_1 - s_i^-, n_2 - s_i^+\rangle, \\ a_1^- |0, n_2\rangle &= 0, \quad a_2^- |n_1, 0\rangle = 0, \quad N_i |n_i\rangle = n_i |n_i\rangle, \end{aligned} \quad (30)$$

where the quantity  $s_i^\pm$  is defined by

$$s_i^\pm = \frac{1}{2}(1 \pm (-)^i).$$

In (30), the quantities

$$F(n_i) = n_i[1 + \kappa(n_i - 1)] \quad i = 1, 2. \quad (31)$$

are the eigenvalues of the operators  $F(N_i)$  given by (27) [22]. They are quadratic with respect to quantum numbers  $n_i$  except the special case  $\kappa = 0$  where usual harmonic oscillators are recovered. In the Schwinger picture, we realize the  $u(2)$  operators as

$$j_+ = a_1^+ a_2^-, \quad j_- = a_1^- a_2^+, \quad j_3 = \frac{1}{2}(N_1 - N_2), \quad j_0 = \frac{1}{2}(N_1 + N_2). \quad (32)$$

Using the expression of the structure function  $F(N)$  (27) and the relations (28), it is simply verified that

$$[j_+, j_-] = 2j_3(I - \kappa + 2\kappa j_0(I + \kappa j_0)) - 4\kappa^2 j_3^3 \quad [j_3, j_\pm] = \pm j_\pm \quad [j_0, j_\alpha] = 0 \quad (33)$$

with  $\alpha = 0, 3, +, -$ . When  $\kappa = 0$ , equation (33) goes back to the commutation relations satisfied by the angular momentum algebra. It is remarkable that the obtained algebra is exactly the cubic Higgs algebra [2] (see also [33]) introduced to establish additional dynamical symmetries for isotropic oscillator and Kepler problem. This shows the relevance the generalized Weyl-Heisenberg algebra  $\mathcal{A}_\kappa$  introduced in [22] in treating such kind of integrable systems.

### 3.2 Fock space

Now, we examine the Hilbert space of the nonlinear algebra satisfying the relations (33). Making use of parallel treatment of angular momentum, it is simple to obtain the unitary representations. Using the actions of the creation and annihilation operators (30), one gets

$$j_+|n_1, n_2\rangle = \sqrt{F(n_1+1)F(n_2)}|n_1+1, n_2-1\rangle, \quad j_-|n_1, n_2\rangle = \sqrt{F(n_1)F(n_2+1)}|n_1-1, n_2+1\rangle \quad (34)$$

where the function  $F(n)$  should be positive. Henceforth,  $\kappa \geq 0$  guarantees the positivity of  $F(n)$  for any integer  $n \in \mathbb{N}$ . However, for  $\kappa < 0$ , the function  $F(n)$  is positive only when  $n = 0, 1, \dots, 1 - 1/\kappa$ . For simplicity, we shall assume that  $-1/\kappa \in \mathbb{N}^*$ . Subsequently, for  $\kappa \geq 0$ , the Fock space  $\mathcal{F}$  is infinite dimensional:

$$\mathcal{F}_+ \equiv \mathcal{F}_{\kappa \geq 0} = \{|n_1, n_2\rangle, n_1 \in \mathbb{N}, n_2 \in \mathbb{N}\}.$$

For  $\kappa < 0$ , the admissible values of the integers  $n_1$  and  $n_2$  are  $0, 1, \dots, d-1$  with  $d = 1 - 1/\kappa$ . In this case, the corresponding Fock space is

$$\mathcal{F}_- \equiv \mathcal{F}_{\kappa < 0} = \{|n_1, n_2\rangle, n_1 = 0, 1, \dots, d-1; n_2 = 0, 1, \dots, d-1\},$$

and therefore its dimension is  $d^2$ . Here again, to pass from the Fock number states to angular momentum basis, we introduce the quantum numbers  $j$  and  $m$  defined as

$$j = \frac{1}{2}(n_1 + n_2) \quad m = \frac{1}{2}(n_1 - n_2),$$

and we set the following correspondence

$$|n_1, n_2\rangle \equiv \left| \frac{n_1 + n_2}{2}, \frac{n_1 - n_2}{2} \right\rangle = |j, m\rangle.$$

With this mapping, the Fock spaces  $\mathcal{F}_+$  and  $\mathcal{F}_-$  decompose as

$$\mathcal{F}_+ = \bigoplus_{j=0}^{\infty} \mathcal{F}_j = \bigoplus_{j=0}^{\infty} \{|j, m\rangle, m = -j, -j+1, \dots, j-1, +j\}$$

and

$$\mathcal{F}_- = \bigoplus_{j=0}^d \mathcal{F}_j = \bigoplus_{j=0}^{d-1} \{|j, m\rangle, m = -j, -j+1, \dots, j-1, +j\}.$$

Clearly, for  $\kappa < 0$ , the Schwinger realization produces only angular momentum with  $j = 0, \frac{1}{2}, 1, \dots, d-1$  so

that the dimension of representations space  $\mathcal{F}_-$  is  $d^2$  ( $\dim \mathcal{F}_- = d^2$ ). This constitutes the main difference with the case where  $\kappa \geq 0$  where the full set of  $su(2)$  unitary representations are recovered.

### 3.3 Fock-Bargmann realization and Coherent states

Fock-Bargmann representation, based on coherent states formalism, is a powerful method for obtaining closed analytic expressions for various properties in quantum systems. We notice that coherent states



for some quantum systems described by nonlinear  $su(2)$  and its noncompact counterpart  $su(1,1)$  algebra are derived in [37, 38, 39, 40]. For the Higgs algebra defined by (33), we shall construct a suitable analytical representation for the states belonging to the subspace  $\mathcal{F}_j = \{|j, m\rangle, m = -j, -j+1, \dots, j-1, +j\}$ . To simplify our notations, we set

$$|j, m\rangle = |n\rangle \quad \text{with} \quad n = j + m.$$

From (34), the actions of the generalized  $su(2)$  algebra operators rewrites

$$j_-|n\rangle = \sqrt{f(n)}|n-1\rangle \quad j_-|n\rangle = \sqrt{f(n+1)}|n+1\rangle \quad j_3|n\rangle = (n-j)|n+1\rangle \quad (35)$$

where the new function  $f(n)$  is defined by

$$f(n) = F(n)F(2j-n+1) = n(2j+1-n)(1+\kappa(n-1))(1+\kappa(2j-n)) \quad (36)$$

in terms of the structure function of the generalized Weyl-Heisenberg algebra  $\mathcal{A}_\kappa$  given by (31). For a representation characterized by the spin  $j$ , the coherent states are of the form

$$|z\rangle = \sum_{n=0}^{2j} a_n z^n |n\rangle \quad (37)$$

where  $z$  is a complex variable,  $n = j + m$  and the  $a_n$  coefficients to be determined. In the Bargmann realization any vector is realized as follows

$$|n\rangle \longrightarrow a_n z^n \equiv \langle \bar{z} | n \rangle \quad (38)$$

and the operator  $j_-$  is assumed acts as a derivation with respect to the complex variable  $z$

$$j_- \longrightarrow \frac{d}{dz}. \quad (39)$$

Thus, using the expression of the action of  $j_-$  (35) and the Bargmann correspondence (38), one shows that the coefficients  $a_n$  satisfy the recursion relation

$$n a_n = \sqrt{f(n)} a_{n-1} \quad (40)$$

where  $f(n)$  is given by (36). This yields

$$a_n = a_0 \frac{\sqrt{f(n)!}}{n!} \quad (41)$$

where  $f(n)! = f(1)f(2)\cdots f(n)$  with  $f(0)! = 1$ . The coefficient  $a_0$  is obtained by imposing the normalization condition for the states  $|z\rangle$ . Finally, one gets

$$|z\rangle = \mathcal{N}^{-1} \sum_{n=0}^{2j} \frac{\sqrt{f(n)!}}{n!} z^n |n\rangle \quad (42)$$

where the  $\mathcal{N}$  normalization factor is given by the sum

$$|\mathcal{N}|^2 = \sum_{n=0}^{2j} \frac{f(n)!}{(n!)^2} |z|^{2n}. \quad (43)$$

In the limiting case  $\kappa \rightarrow 0$ , one recovers the  $su(2)$  coherent states and the standard Bargmann realization based on spin coherent states. In this realization, the operators  $j_3$  and  $j_+$  act, respectively, as follows

$$j_3 \longrightarrow z \frac{d}{dz} - j \quad j_+ \longrightarrow z \left( 1 + \kappa z \frac{d}{dz} \right) \left( 2j - z \frac{d}{dz} \right) \left( 1 + (2j - 1)\kappa - \kappa z \frac{d}{dz} \right). \quad (44)$$

Any state  $|\Psi\rangle$ , belonging to angular momentum space  $\mathcal{F}_j$ ,

$$|\Psi\rangle = \sum_{n=0}^{2j} \Psi_n |n\rangle$$

is represented by an analytical function as follows

$$|\Psi\rangle \longrightarrow \Psi(z) = \mathcal{N} \langle \bar{z} | \Psi \rangle = \sum_{n=0}^{2j} \Psi_n f_n(z) \quad (45)$$

where the monomials  $f_n(z)$  are the analytical functions associated with the vectors  $n$

$$f_n(z) = \mathcal{N} \langle \bar{z} | 2n \rangle = \frac{\sqrt{f(n)!}}{n!} z^n. \quad (46)$$

Using the realization (39) and the results (44), one gets

$$j_3 f_n(z) = (n - j) f_n(z), \quad j_+ f_n(z) = \sqrt{f(n+1)} f_{n+1}(z), \quad j_- f_n(z) = \sqrt{f(n)} f_{n-1}(z), \quad (47)$$

to be compared with (35).

## 4 Concluding remarks

To summarize, we introduced a new variant of polynomial  $u(2)$  algebras. The corresponding unitary representations were constructed. We have shown that the dimension of the corresponding Hilbert space can be modified for some specific values of the parameters  $\kappa_1, \kappa_2, \dots, \kappa_r$ . Via a contraction procedure to the multi-parametric polynomial Weyl-Heisenberg algebra  $\mathcal{A}_{\{\kappa\}}$  introduced in [24] is obtained. The second facet of this letter concerned the realization of Higgs algebra by means of a pair of nonlinear (quadratic) boson generating the one parameter algebra  $\mathcal{A}_\kappa$ . As a further prolongation, it would be interesting to extend the nonlinear bosonic Schwinger realization for other Lie algebras. Another interesting issue concerns other polynomial  $su(2)$  extensions as for instance one defined by the following commutation relations

$$[J_+, J_-] = P(J_3), \quad [J_3, J_+] = G(J_3) J_+, \quad [J_3, J_-] = -J_- G(J_3)$$

where  $P$  and  $G$  are polynomials of the diagonal operator  $J_3$ .

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