Multipartite quantum correlations in even and odd spin coherent states

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Abstract

The key ingredient of the approach, presented in this paper, is the factorization property of $SU(2)$ coherent states upon splitting or decay of a quantum spin system. In this picture, the even and odd spin coherent states are viewed as comprising two, three or more spin subsystems. From this perspective, we investigate the multipartite quantum correlations defined as the sum of the correlations of all possible bi-partitions. The pairwise quantum correlations are quantified by entanglement of formation and quantum discord. A special attention is devoted to tripartite splitting schemes. We explicitly derive the sum of entanglement of formation for all possible bi-partitions. It coincides with the sum of all possible pairwise quantum discord. The conservation relation between the distribution of entanglement of formation and quantum discord, in the tripartite splitting scheme, is discussed. We show that the entanglement of formation and quantum discord possess the monogamy property for even spin coherent states, contrarily to odd ones which violate the monogamy relation when the the overlap of the coherent states approaches the unity.

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1 Introduction

The characterization of nonclassical correlations and nonlocal correlations constitutes one of the main issues intensively investigated in the field of quantum information science. The primary goal is to provide the best way to understand the differences between quantum and classical physics. Quantum correlations constitute a relevant resource to manage information in several ways [1, 2, 3]. Different forms of measures to quantify the degree of quantumness in a multipartite quantum system were introduced. In particular, entanglement of formation has been successfully employed in this sense. However, this measure does not account for all nonclassical aspects of correlations and unentangled mixed states can possess quantum correlations. In this respect, other measures beyond entanglement were proposed in the literature like for instance quantum discord introduced in [4, 5]. It is defined as the difference between the total correlation and classical correlation present in a bipartite system. The quantum discord coincides with entanglement for pure states. For mixed states, the explicit evaluation of quantum discord involves an optimization procedure which is in general a difficult task to achieve. There are few two qubit systems [6, 7, 8, 9, 10, 11, 12] for which analytical results were obtained. To overcome the difficulty in evaluating analytically quantum discord, a geometric method was introduced in [13]. Nowadays, entanglement of formation [14], quantum discord [4, 5] and its geometric variant [13] are typical examples of measures commonly used to decide about the presence of quantum correlations between two different parts composing a bipartite quantum system.

In the recent years, the efforts in identifying and quantifying quantum correlations were extended to correlated nonorthogonal states as for example Glauber coherent states, $SU(2)$ and $SU(1,1)$ coherent states [15, 16] (for a review see [17]). Subsequently, many works have been devoted to investigate their role in quantum cryptography [18], quantum information processing [19] and quantum computing [20, 21, 22]. This is mainly motivated by the possibility to encode quantum information in continuous variables [23]. For example, the even and odd Glauber coherent states, termed also Shrödinger cat states, can be considered as basis states of a logical qubit [24, 25] and provides a practical way to implement experimentally optical quantum systems useful for quantum information.

In other hand, the structure of multipartite quantum systems is a complicated and challenging subject that trigged off a lot of interest during the last decade (see [3] and references therein). In this paper, we shall strictly focus on the study of quantum correlations present in odd and even $SU(2)$ coherent state. In fact, by considering the property according to which a spin-j coherent state $|j,\eta\rangle$ can be factorized as a tensorial product of two $SU(2)$ coherent states $|j_1,\eta\rangle$ and $|j_2,\eta\rangle$ with $(j = j_1 + j_2)$, it is possible to construct a picture where even and odd spin coherent states might be viewed as superpositions of two or more spin coherent systems. The idea of entanglement in a single particle, caused by quantum correlations between its intrinsic degrees of freedom, was discussed in [26, 27, 28]. Consequently, it is seems natural to assume that a odd or even spin-j coherent state presents quantum correlations between its intrinsic parts resulting from the splitting of the spin$j$ into two or more subcomponents. In this scheme, one can analyze the properties of multipartite quantum correlations in many spin systems. The best way to approach this question is the use of bipartite measures. This approach has the advantage relying upon bipartite measures of entanglement of formation and quantum discord that are physically motivated and analytically computable. Also, another important question emerging in this context concerns the limitations of sharing quantum correlations. Indeed, the distribution of quantum correlations among the subsystems of a multipartite quantum system is constrained by the so-called monogamy relation. It was firstly proposed by Coffman, Kundo and Wootters in 2001 [29] in analyzing the distribution of entanglement in a tripartite qubit system. Since then, the monogamy relation was extended to other measures of quantum correlations. Unlike the squared concurrence [29], the entanglement of formation does not satisfy the monogamy relation [29] in a pure tripartite qubit system but it was reported in [30, 31] that it can be satisfied in multimode Gaussian states. Furthermore, quantum correlations, measured by quantum discord, were shown to violate monogamy for some specific quantum states [32, 33, 34, 35, 36]. Now there are many attempts to establish the conditions under which a given quantum correlation measure is monogamous or not. One may quote for instance the results obtained in [37] .

This paper is organized as follows. In Section 2 we give the definitions of the bipartite measures: concurrence, entanglement of formation and quantum discord. We also introduce the measure of multipartite correlations in a given system as the sum of all possible bipartite correlations. Section 3 concerns even and odd spin coherent states. We especially discuss the decomposition property of spin coherent states according to which they split in multipartite spin or qubit systems. In Section 4, we derive the explicit expressions of pairwise quantum correlations present in even and odd spin coherent states decomposed in a pure bipartite system. An appropriate qubit mapping is introduced. The results of section 4 are extended in section 5 to the situation where the spin coherent state splits in three spin sub-systems. A qubit mapping is realized for all possible bi-partitions of the system. The total amount of entanglement of formation is derived in Section 6. Similarly, in Section 7, we explicitly evaluate the total amount of quantum discord present in even and odd spin coherent states viewed as a tripartite system. The sum of pairwise quantum discord is evaluated. It coincides with the total amount of bipartite entanglement of formation in agreement with the result obtained in [38]. This result originates from the conservation relation between the distribution of entanglement of formation and quantum discord proved in [39]. Limitations to sharing entanglement of formation as well as quantum discord are discussed. Some special cases to corroborate our analysis are numerically examined. Concluding remarks close this paper.

2 Quantum correlations

The theoretical investigation of quantum correlations in a multipartite quantum system is motivated by the recent experimental progress in creating and manipulating highly correlated spin ensemble which provide experimentally accessible systems for quantum information processing. In general the analysis of the properties of quantum correlations in many spin systems is difficult. The simply way to approach this problem is the use of bipartite measures that are explicitly computable such as entanglement of formation and usual quantum discord. The definitions of each of these two measures is presented here after. For an arbitrary tripartite state, the quantum correlations present in the system can be computed by considering all possible bipartite splits. The whole system can be partitioned in two different ways. In the first bi-partition scheme, the system splits into two subsystems, one containing one particle and the second comprising the two remaining particles. The second bipartition is obtained by tracing out the degrees of freedom of the third subsystem. In this picture, the total amount of quantum correlations is given by the sum of all possible bipartite quantum correlations.

2.1 Bipartite measures of entanglement of formation and quantum discord

We shall first review briefly the concept of quantum discord [4, 5]. The total correlation is usually quantified by the mutual information, usually expressed in term of von Neumann entropy, as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{1}$$

where ρ_{AB} is the state of a bipartite quantum system composed of the subsystems A and B, the operator $\rho_{A(B)} = \mathrm{Tr}_{B(A)}(\rho_{AB})$ is the reduced state of $A(B)$ and $S(\rho)$ is the von Neumann entropy of a quantum state ρ. The mutual information $I(\rho_{AB})$ contains both quantum and classical correlations. It decomposes as

$$I(\rho_{AB}) = D(\rho_{AB}) + C(\rho_{AB}),$$

and the quantum discord $D(\rho_{AB})$ is defined as the difference between the total correlation $I(\rho_{AB})$ and the classical correlation $C(\rho_{AB})$ present in the bipartite system AB. The classical part $C(\rho_{AB})$ can be determined by optimizing local measurement procedure as follows. Let us consider a von Neumann type measurement, on the subsystem A, belonging to the set one-dimensional projectors $\mathcal{M} = \{M_k\}$ with $\sum_k M_k = \mathbb{I}$. The von Neumann measurement yields the statistical ensemble $\{p_{B,k}, \rho_{B,k}\}$ such that

$$\rho_{AB} \longrightarrow \frac{(M_k \otimes \mathbb{I})\rho_{AB}(M_k \otimes \mathbb{I})}{p_{B,k}}$$

where the measurement operation is written as [40]

$$M_k = U\,\Pi_k\,U^\dagger \tag{2}$$

with $\Pi_k = |k\rangle\langle k|$ $(k = 0, 1)$ is the one dimensional projector for subsystem A along the computational base $|k\rangle$, $U \in SU(2)$ is a unitary operator and

$$p_{B,k} = \mathrm{tr}\Big[(M_k \otimes \mathbb{I})\rho_{AB}(M_k \otimes \mathbb{I})\Big].$$

The amount of information acquired about particle B is then given by

$$S(\rho_B) - \sum_k p_{B,k}\ S(\rho_{B,k}),$$

which depends on measurement $\mathcal{M}$. To remove the measurement dependence, a maximization over all possible measurements is performed and the classical correlation writes

$$\begin{aligned} C(\rho_{AB}) &= \max_{\mathcal{M}} \left[S(\rho_B) - \textstyle\sum_k\ p_{B,k}\ S(\rho_{B,k})\right] \\ &= S(\rho_B) - \widetilde{S}_{\min} \end{aligned} \tag{3}$$

where $\widetilde{S}_{\min}$ denotes the minimal value of the conditional entropy

$$\widetilde{S} = \sum_k p_{B,k}\ S(\rho_{B,k}). \tag{4}$$

When optimization is taken over all perfect measurement, the quantum discord is

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) = S(\rho_A) + \widetilde{S}_{\min} - S(\rho_{AB}). \tag{5}$$

The explicit evaluation of quantum discord (5) requires the analytical computation of $\widetilde{S}_{\min}$. This quantity was explicitly derived only for few exceptional two-qubit quantum states. One may quote for instance the results obtained in [7, 41] (see also [11, 12, 42]). In this paper, we shall mainly concerned with two-rank quantum states for which the minimization of the conditional entropy (4) can be exactly performed by purifying the density matrix ρ_{AB} and making use of Koashi-Winter relation [43] (see also [44]). This relation establishes the connection between the classical correlation of a bipartite state ρ_{AB} and the entanglement of formation of its complement ρ_{BC}. Hereafter, we discuss shortly this method. We assume that the density matrix ρ_{AB} has two non vanishing eigenvalues (two-rank matrix). It decomposes as

$$\rho_{AB} = \lambda_+ |\phi_+\rangle_{AB}\langle\phi_+| + \lambda_- |\phi_-\rangle_{AB}\langle\phi_-| \tag{6}$$

where λ_+ and λ_- are the eignevalues of ρ_{AB} and the corresponding eigenstates are denoted by $|\phi_+\rangle_{AB}$ and $|\phi_-\rangle_{AB}$ respectively. The purification of the mixed state ρ_{AB} is realized by attaching a qubit C to the two-qubit system A and B. This yields

$$|\phi\rangle_{ABC} = \sqrt{\lambda_+}|\phi_+\rangle_{AB} \otimes |\mathbf{0}\rangle_C + \sqrt{\lambda_-}|\phi_-\rangle_{AB} \otimes |\mathbf{1}\rangle_C \tag{7}$$

such that the whole system ABC is described by the pure density matrix $\rho_{ABC} = |\phi\rangle_{ABC}\langle\phi|$ from which one has the bipartite densities $\rho_{AB} = \mathrm{Tr}_C \rho_{ABC}$ and $\rho_{BC} = \mathrm{Tr}_A \rho_{ABC}$. According to Koachi-Winter relation [43], the minimal value of the conditional entropy coincides with the entanglement of formation of ρ_{BC}:

$$\widetilde{S}_{\min} = E(\rho_{BC}) \tag{8}$$

which is given by

$$\widetilde{S}_{\min} = E(\rho_{BC}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |\mathcal{C}(\rho_{BC})|^2}) \tag{9}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and $\mathcal{C}(\rho_{BC})$ is the concurrence of the density matrix ρ_{BC}. We recall that for ρ_{12} the density matrix for a pair of qubits 1 and 2, which may be pure or mixed, the concurrence is [45]

$$\mathcal{C}_{12} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{10}$$

for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ the square roots of the eigenvalues of the "spin-flipped" density matrix

$$\varrho_{12} \equiv \rho_{12}(\sigma_y \otimes \sigma_y)\rho_{12}^{\star}(\sigma_y \otimes \sigma_y), \tag{11}$$

where the star stands for complex conjugation in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ with the Pauli matrix is $\sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$. Nonzero concurrence traduces the entanglement between the qubits 1 and 2, otherwise they are separable. Using the equations (5) and (9), the quantum discord writes as

$$D_{AB} \equiv D_{AB}^{\rightarrow} = S_A - S_{AB} + E_{BC}. \tag{12}$$

In the same manner, when the measurement is performed on the subsystem B, it is simply verified that the quantum discord takes the form

$$D_{BA} \equiv D_{AB}^{\leftarrow} = S_B - S_{AB} + E_{AC} \tag{13}$$

Notice that for a pure density matrix ρ_{AB}, the quantum discord reduces to entanglement of formation $E(\rho_{AB})$.

2.2 Multipartite quantum correlations

The measure of multipartite quantum correlations constitutes an important issue in the context of quantum information. Some attempts to provide a precise way to quantify and characterize the genuine multipartite correlations were discussed in the literature yielding different approaches [38, 46, 47, 48]. In particular, Rulli and Sarandy [48] defined the multipartite measure of quantum correlation as the maximum of the quantum correlations existing between all possible bipartition of the multipartite quantum system. In a similar way, Z-H Ma and coworkers [38] suggested a slightly different definition to quantify the global multipartite quantum correlation. It is defined as the sum of the correlations in all possible bi-partitions. In this paper, paralleling the treatment discussed in [38], we shall quantify the global quantum correlations present in even and odd spin coherent states as follows. For a tripartite spin coherent states system $(j_1j_2j_3)$ arising from the decomposition of a spin-j coherent state with $j = j_1 + j_2 + j_3$, the total amount of quantum correlation is defined by

$$\begin{aligned} Q(j_1, j_2, j_3) &= \frac{1}{12}(Q_{j_1j_2} + Q_{j_2j_1} + Q_{j_1j_3} + Q_{j_3j_1} + Q_{j_2j_3} + Q_{j_3j_2} \\ &+ Q_{j_1(j_2j_3)} + Q_{(j_2j_3)j_1} + Q_{j_2(j_1j_3)} + Q_{(j_1j_3)j_2} + Q_{j_3(j_1j_2)} + Q_{(j_1j_2)j_3}) \end{aligned} \tag{14}$$

where the bipartite measure Q stands for entanglement of formation or quantum discord. More details concerning the remarkable splitting property of spin coherent states will be presented in the next section. In other hand, as we shall deal with tripartite quantum states, it is natural to investigate the intriguing monogamy relation of quantum correlation present in spin coherent coherent states. The concept of monogamy can be introduced as follows. Let $Q_{A|B}$ denotes the shared correlation Q between A and B. Similarly, let us denote by $Q_{A|C}$ the measure of the correlation between A and C and $Q_{A|BC}$ the correlation shared between A and the composite subsystem BC comprising B and C. The bipartite measure of correlations Q is monogamous if $Q_{A|BC}$ is greater that the sum of $Q_{A|B}$ and $Q_{A|C}$:

$$Q_{A|BC} \geq Q_{A|B} + Q_{A|C}. \tag{15}$$

This inequality imposes severe limitations to sharing quantum correlations. The monogamy of entanglement of formation and quantum discord in tripartite spin coherent states are examined in the sections 6 and 7. It must be emphasized that the conditions under which any measure of quantum correlations that comprise and go beyond entanglement of formation was discussed by Fanchini et al in [49] for an arbitrary pure tripartite state. In particular, the authors developed an elegant operational approach based on the discrepancy between classical and quantum correlations to set up the constraints that any pure tripartite state must satisfy such that the entanglement of formation follow the monogamy property. This approach allows also to understand the result obtained by Giorgi [32] according to which the entanglement of formation and quantum discord obey the same monogamous relation.

3 Spin coherent states as multi-qubit systems

3.1 Multi-qubit structure of Bloch coherent spin states

An arbitrary spin system is described by the $su(2)$ algebra generated by the operators J_+, J_- and J_3 satisfying the following structure relations

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_-, J_+] = -2J_3 \ . \tag{16}$$

The different irreducible representations classes of the group $SU(2)$ are completely determined by the quantum angular momentum j which may take integer or half integer values ($j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$). The $(2j+1)$-dimensional Hilbert space is spanned by the irreducible tensorial set $\{|j,m\rangle, m = -j, -j+1, \cdots, j-1, j\}$ characterizing the spin-j representations of the group $SU(2)$. The standard $SU(2)$ coherent states are obtained by the action of an element of the coset space $SU(2)/U(1)$

$$D_j(\xi) = \exp(\xi J_+ - \xi^* J_-) \ , \tag{17}$$

on the extremal state $|j,-j\rangle$. This action gives the states

$$|j,\eta\rangle = D_j(\xi)|j,-j\rangle = \exp(\xi J_+ - \xi^* J_-)|j,-j\rangle = (1+|\eta|^2)^{-j}\exp(\eta J_+)|j,-j\rangle \ , \tag{18}$$

where $\eta = (\xi/|\xi|)\tan|\xi|$. In the basis $\{|j,m\rangle\}$, they write

$$|j,\eta\rangle = (1+|\eta|^2)^{-j}\sum_{m=-j}^{j}\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}\eta^{j+m}|j,m\rangle\ . \tag{19}$$

They satisfy the resolution to identity property

$$\int d\mu(j,\eta)|j,\eta\rangle\langle j,\eta| = I\ , \qquad d\mu(j,\eta) = \frac{2j+1}{\pi}\frac{d^2\eta}{(1+|\eta|^2)^2}\ . \tag{20}$$

The spin coherent states are not orthogonal to each other:

$$\langle j,\eta_1|j,\eta_2\rangle = (1+|\eta_1|^2)^{-j}(1+|\eta_2|^2)^{-j}(1+\eta_1^*\eta_2)^{2j}\ . \tag{21}$$

The resolution to identity makes possible to expand an arbitrary state in terms of the coherent states $|j,\eta\rangle$. In the special case $j=\frac{1}{2}$, the spin coherent states (19) reduce to

$$|\eta\rangle = \frac{1}{\sqrt{1+\bar{\eta}\eta}}|\downarrow\rangle + \frac{\eta}{\sqrt{1+\bar{\eta}\eta}}|\uparrow\rangle. \tag{22}$$

Here and in the following $|\eta\rangle$ is short for the spin-$\frac{1}{2}$ coherent state $|\frac{1}{2},\eta\rangle$ with $|\uparrow\rangle \equiv |\frac{1}{2},\frac{1}{2}\rangle$ and $|\downarrow\rangle \equiv |\frac{1}{2},-\frac{1}{2}\rangle$). It is important to notice that the tensorial product of two $SU(2)$ coherent states $|j_1,\eta\rangle$ and $|j_2,\eta\rangle$ produces a spin-(j_1+j_2) coherent state labeled by the same variable:

$$|j_1,\eta\rangle\otimes|j_2,\eta\rangle = (D_{j_1}\otimes D_{j_2})\left(|j_1,j_1\rangle\otimes|j_2,j_2\rangle\right) = D_{j_1+j_2}\,|j_1+j_2,j_1+j_2\rangle\ =\ |j_1+j_2,\eta\rangle. \tag{23}$$

Only coherent states possess this remarkable property. It allows to write any spin-j coherent states as a $2j$ tensorial product of spin-$\frac{1}{2}$ coherent states:

$$|j,\eta\rangle = (|\eta\rangle)^{\otimes 2j} = \left(\frac{1}{\sqrt{1+\bar{\eta}\eta}}|\downarrow\rangle + \frac{\eta}{\sqrt{1+\bar{\eta}\eta}}|\uparrow\rangle\right)^{\otimes 2j} = (1+\bar{\eta}\eta)^{-j}\sum_{m=-j}^{+j}\begin{pmatrix}2j\\ j+m\end{pmatrix}^{\frac{1}{2}}\eta^{j+m}|j,m\rangle,$$

reflecting that a spin-j coherent state may be viewed as a multipartite state containing $2j$ qubits.

3.2 Even and odd coherent states

The even and odd spin coherent states are defined by

$$|j,\eta,m\rangle = \mathcal{N}_m(|j,\eta\rangle + e^{im\pi}|j,-\eta\rangle) \tag{24}$$

where the integer $m\in\mathbb{Z}$ takes the values $m=0$ (mod 2) and $m=1$ (mod 2). The normalization factor $\mathcal{N}_m$ is

$$\mathcal{N}_m = \left[2+2p^{2j}\cos m\pi\right]^{-1/2}$$

where p denotes the overlap between the states $|\eta\rangle$ and $|-\eta\rangle$. It is given

$$p = \langle\eta|-\eta\rangle = \frac{1-\bar{\eta}\eta}{1+\bar{\eta}\eta}. \tag{25}$$

For $j = \frac{1}{2}$, the even and odd coherent states coincide with $|\uparrow\rangle$ and $|\downarrow\rangle$. They can be identified with basis states for a logical qubit as $|0\rangle \rightarrow |\uparrow\rangle$ and $|1\rangle \rightarrow |\downarrow\rangle$. This line of reasoning can be extended to higher spin values and provides scheme to encode information in superpositions of arbitrary spin coherent states, especially even and odd ones. Indeed, the states $|j,\eta,0\rangle$ and $|j,\eta,1\rangle$ define a two-dimensional orthogonal basis and give a first possible encoding scheme. Thus, one can identify the even state $|j,\eta,0\rangle$ and the odd state $|j,\eta,1\rangle$ as basis of a logical qubit as

$$|j,\eta,0\rangle \longrightarrow |0\rangle_j \qquad |j,\eta,1\rangle \longrightarrow |1\rangle_j.$$

Others encoding schemes involving more qubits are also possible. They can be realized using the factorization or the splitting property of spin coherent states (23). In fact, the states (24) can be also expressed as

$$|j,\eta,m\rangle = \mathcal{N}_m(|j_1,\eta\rangle \otimes |j_2,\eta\rangle + e^{im\pi}|j_1,-\eta\rangle \otimes |j_2,-\eta\rangle) \tag{26}$$

with $j = j_1 + j_2$. They can rewritten as a two qubit states in the basis

$$|j_i,\eta,0\rangle \longrightarrow |0\rangle_{j_i} \qquad |j_i,\eta,1\rangle \longrightarrow |1\rangle_{j_i}, \quad i = 1,2.$$

defined by means of odd and even spin coherent associated with the angular momenta j_1 and j_2. This construction is easily generalizable to three and more qubits. In this manner, the states $|j,\eta,m\rangle$ can be viewed as multipartite fermionic coherent states:

$$|j,\eta,m\rangle = \mathcal{N}_m((|\eta\rangle)^{\otimes 2j} + e^{im\pi}\,(|-\eta\rangle)^{\otimes 2j}). \tag{27}$$

Furthermore, the logical qubits $|j,\eta,0\rangle$ (even) and $|j,\eta,1\rangle$ (odd) spin coherent states behave like a multipartite state of Greenberger-Horne-Zeilinger (GHZ) type [50] in the asymptotic limit $p \rightarrow 0$. In this special limiting case, the states $|\eta\rangle$ and $|-\eta\rangle$ approach orthogonality and an orthogonal basis can be defined such that $|\mathbf{0}\rangle \equiv |\eta\rangle$ and $|\mathbf{1}\rangle \equiv |-\eta\rangle$. Thus, the state $|j,\eta,m\rangle$ becomes of GHZ-type

$$|j,\eta,m\rangle \sim |\mathrm{GHZ}\rangle_{2j} = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \cdots \otimes |\mathbf{0}\rangle + e^{im\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes \cdots \otimes |\mathbf{1}\rangle). \tag{28}$$

The second limiting case corresponds to the situation when $p \rightarrow 1$ (or $\eta \rightarrow 0$). In this case it is simple to check that the state $|j,\eta,m = 0 \pmod 2\rangle$ (27) reduces to ground state of a collection of $2j$ fermions

$$|j,0,0 \pmod 2\rangle \sim |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \tag{29}$$

and the state $|j,\eta,1 \pmod 2\rangle$ becomes a multipartite state of W type [51]

$$|j,0,1 \pmod 2\rangle \sim |\mathrm{W}\rangle_{2j} = \frac{1}{\sqrt{2j}}(|\uparrow\rangle\otimes|\downarrow\rangle\otimes\cdots\otimes|\downarrow\rangle+|\downarrow\rangle\otimes|\uparrow\rangle\otimes\ldots\otimes|\downarrow\rangle+\cdots+|\downarrow\rangle\otimes|\downarrow\rangle\otimes\cdots\otimes|\uparrow\rangle)\,. \tag{30}$$

The even spin coherent states $|j,\eta,m = 0 \pmod 2\rangle$ interpolate continuously between GHZ_{2j} states ($p \rightarrow 0$) and the completely separable state $|\downarrow\rangle\otimes|\downarrow\rangle\otimes\cdots\otimes|\downarrow\rangle$ ($p \rightarrow 1$). In the odd case, corresponding to $|j,\eta,m = 1 \pmod 2\rangle$, we obtain states interpolating between states of GHZ_{2j} type ($p \rightarrow 0$) and states of W_{2j} type ($p \rightarrow 1$).

The decomposition property (23) provides us with a picture where even and odd spin coherent states can be considered as comprising multipartite spin subsystems. This is our main motivation to investigate the quantum correlations present in a single spin coherent state. This issue is discussed in what follows.

4 Bipartite splitting and bipartite correlations

In this section, we first discuss the bipartite splitting described by the equation (26). In this scheme, the entire system contains two subsystems characterized by the angular momenta j_1 and j_2 such that $j = j_1 + j_2$. Accordingly, $(2j-1)$ possible bipartite splitting are possible:

$$j_1 = j - \frac{s}{2} \quad j_2 = \frac{s}{2} \quad s = 1, 2, \cdots, 2j-2, 2j-1,$$

and subsequently it is interesting to compare the pairwise quantum correlations in each possible bipartite splitting.

4.1 Bipartite entanglement of formation

As discussed in the previous section, for each bipartition s ($s = 1, 2, \cdots, 2j-1$), the coherent state $|j, \eta, m\rangle$ can be expressed as a state of two logical qubits. In this sense, for each subsystem, an orthogonal basis $\{|0\rangle_l, |1\rangle_l\}$, with $l = j_1$ or j_2, can be defined as

$$|0\rangle_l = \frac{|l, \eta\rangle + |l, -\eta\rangle}{\sqrt{2(1+p^{2l})}} \qquad |1\rangle_l = \frac{|l, \eta\rangle - |l, -\eta\rangle}{\sqrt{2(1-p^{2l})}}. \tag{31}$$

The bipartite density matrix $\rho = |j, \eta, m\rangle\langle j, \eta, m|$ is pure. In this situation, the quantum discord for the pure state $\rho_{AB} \equiv \rho$ coincides with the entanglement of formation. It is given by the von Neumann entropy of the subsystem characterized by the spin j_1:

$$D(\rho) = E(\rho) = S(\rho_{j_1}) \tag{32}$$

where $\rho_{j_1} = \mathrm{Tr}_{j_2}(\rho)$ is the reduced density matrix of the first subsystem obtained by tracing out the spin j_2 . Thus, the quantum discord writes as

$$D(\rho) = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_- \tag{33}$$

in term of the eigenvalues of the reduced density matrix ρ_{j_1} given by

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - \mathcal{C}^2}\right). \tag{34}$$

In Eq.(34), $\mathcal{C}$ is the concurrence between the two subsystems given by

$$\mathcal{C} = \frac{\sqrt{1-p^{4j_1}}\sqrt{1-p^{4j_2}}}{1+p^{2j}\cos m\pi} \tag{35}$$

that is simply obtained by using the qubit mapping (31). It follows that the entanglement of formation writes

$$E_{j_1,j_2} \equiv E(\rho) = H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{2j_1} + p^{2j_2}\cos m\pi}{1 + p^{2j}\cos m\pi}\right), \tag{36}$$

where H stands for the binary entropy defined above. Notice that the entanglement of formation satisfies the symmetry relation

$$E_{j_1,j_2} = E_{j_2,j_1} \tag{37}$$

as expected. For $p \to 0$, the state (26) reduces to a bipartite state of GHZ type which is maximally entangled ($\mathcal{C} = 1$) and the entanglement of formation is $E(\rho) = 1$. The limiting case $p \to 1$ is slightly different. In fact, we have $E(\rho) = 0$ for m even (i.e. symmetric pure states). The odd spin coherent states (i.e. m odd) become of W type when $p \to 1$ and the bipartite concurrence writes

$$\mathcal{C} = 2\,\frac{\sqrt{j_1 j_2}}{j_1 + j_2}.$$

It follows that the corresponding pairwise quantum entanglement takes the form

$$E(\rho) = D(\rho) = H\left(\frac{1}{2} + \frac{1}{2}\frac{j_1 - j_2}{j_1 + j_2}\right).$$

The entanglement of formation in W states is maximal when $j_1 = j_2$ ($E(\rho) = 1$). In other hand, in a splitting scheme such as $j_2 \ll j_1$ or $j_1 \ll j_2$, the states of W type are unentangled ($E(\rho) = 0$).

4.2 Illustration

To exemplify the above results, we consider the even and odd coherent states associated with the spin $j = 2$. The three possible bipartite splitting schemes are

$$(j_1 = \frac{3}{2}, j_2 = \frac{1}{2}) \quad (j_1 = 1, j_2 = 1) \quad (j_1 = \frac{1}{2}, j_2 = \frac{3}{2})$$

Using the equation (36) and the relation (37), one gets

$$E_{\frac{3}{2},\frac{1}{2}} = E_{\frac{1}{2},\frac{3}{2}} = H\left(\frac{1}{2} + \frac{p}{2}\frac{p^2 + \cos m\pi}{1 + p^4\cos m\pi}\right) \tag{38}$$

and

$$E_{1,1} = H\left(\frac{1}{2} + \frac{p^2}{2}\frac{1 + \cos m\pi}{1 + p^4\cos m\pi}\right) \tag{39}$$

The behavior of the entanglement of formation $E_{\frac{3}{2},\frac{1}{2}}$ and $E_{1,1}$ versus the overlap p is plotted in the figures 1 and 2 corresponding respectively to even ($m = 0$) and odd ($m = 1$) spin coherent states. As seen from the figures, in both cases the entanglement of formation in the splitting scheme $2 \longrightarrow (1,1)$ is greater than one existing between the spin subsystems arising from the decomposition $2 \longrightarrow (\frac{3}{2}, \frac{1}{2})$ for any value of p. In general, for a given spin j, the maximal value of entanglement of formation E_{j_1,j_2} is reached in the bipartition where $j_1 = j_2 = \frac{j}{2}$. In figure 2, for odd spin coherent states, we have $E_{1,1} = 1$ as it can be verified from the expression (39).

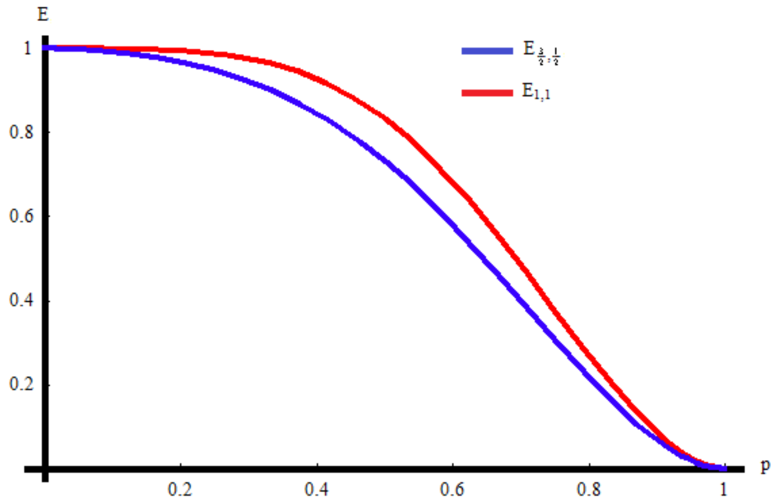

FIG. 1: The pairwise entanglement of formation $E = E_{j_1,j_2}$ versus the overlap p for $(j_1 = \frac{3}{2}, j_2 = \frac{1}{2})$ and $(j_1 = 1, j_2 = 1)$ with $m = 0$.

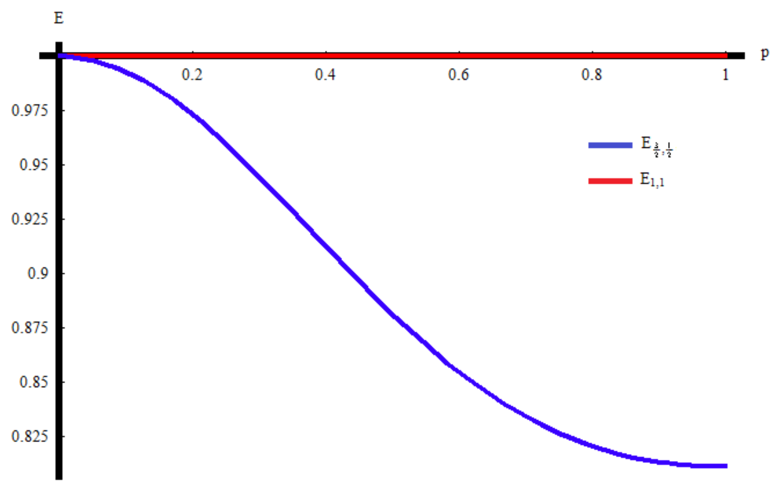

FIG. 2: The pairwise entanglement of formation $E = E_{j_1,j_2}$ versus the overlap p for $(j_1 = \frac{3}{2}, j_2 = \frac{1}{2})$ and $(j_1 = 1, j_2 = 1)$ with $m = 1$.

5 Three modes splitting and qubit mapping

Analogously to the bipartite case, we consider in this section the tripartite splitting of even and odd spin coherent states (24). The entire system decays into three subsystems, one subsystem describing a particle of spin j_1, the second refers to a particle of spin j_2 and the remaining particle is of spin $j_3 = j - j_1 - j_2$. In this scheme, the state $|j, \eta, m\rangle$ writes as

$$|j, \eta, m\rangle = \mathcal{N}_m(|j_1, \eta\rangle \otimes |j_2, \eta\rangle \otimes |j_3, \eta\rangle + e^{im\pi}|j_1, -\eta\rangle \otimes |j_2, -\eta\rangle \otimes |j_3, -\eta\rangle). \tag{40}$$

To evaluate the bipartite quantum correlations present coherent states decomposed as in (40), two different bi-partitions are considered. The first one yields pure bipartite states and the second one involves mixed two-qubit states.

5.1 Bipartite pure states

The pure bi-partitions of the state (40) can be introduced in three different ways. In the first one, the state $|j,\eta,m\rangle$ is written as

$$|j,\eta,m\rangle_{j_1|j-j_1} = \mathcal{N}_m(|j_1,\eta\rangle \otimes |j-j_1,\eta\rangle + e^{im\pi}|j_1,-\eta\rangle \otimes |j-j_1,-\eta\rangle). \tag{41}$$

Similarly, the state (40) can be also partitioned as

$$|j,\eta,m\rangle_{j_2|j-j_2} = \mathcal{N}_m(|j_2,\eta\rangle \otimes |j-j_2,\eta\rangle + e^{im\pi}|j_2,-\eta\rangle \otimes |j-j_2,-\eta\rangle). \tag{42}$$

The third bipartition is given by

$$|j,\eta,m\rangle_{j_3|j-j_3} = \mathcal{N}_m(|j_3,\eta\rangle \otimes |j-j_3,\eta\rangle + e^{im\pi}|j_3,-\eta\rangle \otimes |j-j_3,-\eta\rangle). \tag{43}$$

For each bipartition, the state $|j,\eta,m\rangle$ can be converted into a state of two logical qubits. This is achieved by introducing, for the first subsystem, the orthogonal basis $\{|0\rangle_l, |1\rangle_l\}$, with $l = j_1, j_2$ or j_3, defined as

$$|0\rangle_l = \frac{|l,\eta\rangle + |l,-\eta\rangle}{\sqrt{2(1+p^{2l})}} \qquad |1\rangle_l = \frac{|l,\eta\rangle - |l,-\eta\rangle}{\sqrt{2(1-p^{2l})}}, \tag{44}$$

and, for the second subsystem, the orthogonal basis $\{|0\rangle_{j-l}, |1\rangle_{j-l}\}$ given by

$$|0\rangle_{j-l} = \frac{|j-l,\eta\rangle + |j-l,-\eta\rangle}{\sqrt{2(1+p^{2(j-l)})}} \qquad |1\rangle_{j-l} = \frac{|j-l,\eta\rangle - |j-l,-\eta\rangle}{\sqrt{2(1-p^{2(j-l)})}}. \tag{45}$$

Reporting the equations (44) and (45) in (41), (42) and (43), one has the expression of the pure state $|j,\eta,m\rangle_{l|j-l}$ in the basis $\{|0\rangle_l \otimes |0\rangle_{j-l}, |0\rangle_l \otimes |1\rangle_{j-l}, |1\rangle_l \otimes |0\rangle_{j-l}, |1\rangle_l \otimes |1\rangle_{j-l}\}$. It is given by

$$|j,\eta,m\rangle_{l|j-l} = \sum_{\alpha=0,1}\sum_{\beta=0,1} C_{\alpha,\beta}|\alpha\rangle_l \otimes |\beta\rangle_{j-l} \tag{46}$$

where the coefficients $C_{\alpha,\beta}$ are

$$C_{0,0} = \mathcal{N}_m(1+e^{im\pi})a_l a_{j-l}, \qquad C_{0,1} = \mathcal{N}_m(1-e^{im\pi})a_l b_{j-l}$$

$$C_{1,0} = \mathcal{N}_m(1-e^{im\pi})a_{j-l} b_l, \qquad C_{1,1} = \mathcal{N}_m(1+e^{im\pi})b_l b_{j-l}.$$

in terms of the quantities

$$a_k = \sqrt{\frac{1+p^{2k}}{2}}, \qquad b_k = \sqrt{\frac{1-p^{2k}}{2}} \qquad \text{for } k = l, j-l$$

involving the overlap p (25) which is related to the non-orthogonality of two spin coherent states of equal amplitude and opposite phase.

5.2 Bipartite mixed states

The second class of bipartite density matrices can be realized from the state (40) by considering the reduced density matrices $\rho_{l_1 l_2}$ that are obtained by tracing out the degrees of freedom of the third subsystem. There are three different density matrices $\rho_{j_1 j_2}$, $\rho_{j_2 j_3}$ and $\rho_{j_1 j_3}$. Explicitly, they are given by

$$\begin{aligned} \rho_{l_1 l_2} &= \mathrm{Tr}_{l_3}(|j,\eta,m\rangle\langle j,\eta,m|) \\ &= \mathcal{N}_m^2(|\eta,\eta)(\eta,\eta| + |-\eta,-\eta)(-\eta,-\eta| + e^{im\pi}q|-\eta,-\eta)(\eta,\eta| + e^{-im\pi}q|\eta,\eta)(-\eta,-\eta|) \end{aligned} \quad (47)$$

with $q \equiv p^{2(j-l_1-l_2)} = p^{2l_3}$ and

$$|\pm\eta,\pm\eta) = |l_1,\pm\eta\rangle \otimes |l_2,\pm\eta\rangle.$$

It is interesting to note that the density matrix $\rho_{l_1 l_2}$ is a two-rank operator. Indeed, it rewrites as

$$\rho_{l_1 l_2} = \frac{1}{2}(1+q)\,\frac{\mathcal{N}_m^2}{\mathcal{N}_+^2}\,|\phi_+\rangle\langle\phi_+| + \frac{1}{2}(1-q)\,\frac{\mathcal{N}_m^2}{\mathcal{N}_-^2}\,|\phi_-\rangle,\langle\phi_-| \quad (48)$$

where

$$|\phi_\pm\rangle = \mathcal{N}_\pm(|l_1,\eta\rangle \otimes |l_2,\eta\rangle \pm e^{im\pi}|l_1,-\eta\rangle \otimes |l_2,-\eta\rangle)$$

and

$$\mathcal{N}_\pm^2 = 2 \pm 2p^{2(l_1+l_2)}\cos m\pi.$$

In this case, the density matrix $\rho_{l_1 l_2}$ can be also converted into a two-qubit system by an appropriate qubit mapping. For this, we introduce an orthogonal pair $\{|0\rangle_l, |1\rangle_l\}$ as

$$|0\rangle_l = \frac{|l,\eta\rangle + |l,-\eta\rangle}{\sqrt{2(1+p^{2l})}} \qquad |1\rangle_l = \frac{|l,\eta\rangle - |l,-\eta\rangle}{\sqrt{2(1-p^{2l})}}. \quad (49)$$

where $l = l_1$ for the first subsystem and $l = l_2$ for the second. Substituting the equation (49) into (47), we obtain the density matrix

$$\rho_{l_1 l_2} = \mathcal{N}^2 \begin{pmatrix} 2a_1^2a_2^2(1+q\cos m\pi) & 0 & 0 & 2a_1b_1a_2b_2(1+q\cos m\pi) \\ 0 & 2a_1^2b_2^2(1-q\cos m\pi) & 2a_1b_1a_2b_2(1-q\cos m\pi) & 0 \\ 0 & 2a_1b_1a_2b_2(1-q\cos m\pi) & 2a_2^2b_1^2(1-q\cos m\pi) & 0 \\ 2a_1b_1a_2b_2(1+q\cos m\pi) & 0 & 0 & 2b_1^2b_2^2(1+q\cos m\pi) \end{pmatrix} \quad (50)$$

in the basis $\{|0_{l_1},0_{l_2}\rangle, |0_{l_1},1_{l_2}\rangle, |1_{l_1},0_{l_2}\rangle, |1_{l_1},1_{l_2}\rangle\}$ where the quantities a_1, b_1, a_2, b_2 are defined by

$$a_i = \sqrt{\frac{1+p^{2l_i}}{2}}, \qquad b_i = \sqrt{\frac{1-p^{2l_i}}{2}} \qquad \text{for } i = 1,2$$

6 Quantum entanglement in the three splitting scheme

6.1 Entanglement of formation

In the pure bipartite splitting scheme, the concurrence is given by

$$\mathcal{C}(\rho_{k_1|k_2k_3}) == \frac{\sqrt{1-p^{4k_1}}\sqrt{1-p^{4(j-k_1)}}}{1+p^{2j}\cos m\pi} \quad (51)$$

where the triplet (k_1, k_2, k_3) stands for (j_1, j_2, j_3) , (j_2, j_1, j_3) and (j_3, j_1, j_2) corresponding respectively to the states (41), (42) and (43). Subsequently, the entanglement of formation writes

$$E(\rho_{k_1|k_2k_3}) = H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{2k_1} + p^{2(j-k_1)}\cos m\pi}{1 + p^{2j}\cos m\pi}\right). \tag{52}$$

For mixed bipartite states belonging to the second bi-partitioning class (47), the concurrence is given by

$$\mathcal{C}(\rho_{l_1l_2}) = p^{2(j-l_1-l_2)}\ \frac{\sqrt{(1-p^{4l_1})(1-p^{4l_2})}}{1 + p^{2j}\cos m\pi} \tag{53}$$

where the reduced density matrix $\rho_{l_1l_2}$ stands for $\rho_{j_1j_2}$, $\rho_{j_2j_3}$ and $\rho_{j_1j_3}$. The entanglement of formation writes

$$E(\rho_{l_1l_2}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{p^{4(j-l_1-l_2)}(1-p^{4l_1})(1-p^{4l_2})}{(1 + p^{2j}\cos m\pi)^2}}\right). \tag{54}$$

6.2 Multipartite entanglement of formation

When the bipartite quantum correlations are quantified by the entanglement of formation, the definition (14) gives

$$E(j_1, j_2, j_3) = \frac{1}{6}(E(\rho_{j_1j_2}) + E(\rho_{j_1j_3}) + E(\rho_{j_2j_3}) + E(\rho_{j_1|j_2j_3}) + E(\rho_{j_2|j_1j_3}) + E(\rho_{j_3|j_1j_2})) \tag{55}$$

Using the results (52) and (54), the total amount of quantum entanglement is explicitly given by

$$\begin{aligned} E(j_1, j_2, j_3) &= \frac{1}{6}\Bigg[H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{2j_1} + p^{2(j_2+j_3)}\cos m\pi}{1 + p^{2j}\cos m\pi}\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{p^{4j_1}(1-p^{4j_2})(1-p^{4j_3})}{(1 + p^{2j}\cos m\pi)^2}}\right) \\ &+ H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{2j_2} + p^{2(j_1+j_3)}\cos m\pi}{1 + p^{2j}\cos m\pi}\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{p^{4j_2}(1-p^{4j_1})(1-p^{4j_3})}{(1 + p^{2j}\cos m\pi)^2}}\right) \\ &+ H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{2j_3} + p^{2(j_1+j_2)}\cos m\pi}{1 + p^{2j}\cos m\pi}\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{p^{4j_3}(1-p^{4j_1})(1-p^{4j_2})}{(1 + p^{2j}\cos m\pi)^2}}\right)\Bigg] \end{aligned} \tag{56}$$

which is completely symmetric in j_1, j_2 and j_3. This quantity will be compared with the sum of pairwise quantum discord of all possible bi-partitions of the state (40) and its behavior in terms of the overlap p in some particular cases is examined in Section 7.

6.3 Monogamy of entanglement of formation

The entanglement shared by more than two parties constitutes a subtle issue in investigating multipartite correlations. Thus, considering the limitations of sharing entanglement in the orthogonal case, we study the monogamy of entanglement of formation in tripartite spin coherent states. In this respect, we analyze the situations where the following inequality

$$E(\rho_{l_1l_2}) + E(\rho_{l_1l_3}) \leq E(\rho_{l_1|l_2l_3})$$

is satisfied or violated. The notations are as above. Clearly, to decide if the entanglement of formation is monogamous or not in spin coherent states, we shall treat some particular cases. We first consider the splitting $j_1 = j_2 = j_3 = \frac{1}{2}$ which arises from the decomposition of even and odd coherent states associated with the spin $j = \frac{3}{2}$. The behavior of the entanglement of formation difference :

$$\Delta E = E(\rho_{j_1|j_2 j_3}) - E(\rho_{j_1 j_2}) - E(\rho_{j_1 j_2}),$$

for even and odd spin coherent states, are reported in the figure 3. They show that the entanglement of formation satisfies always the monogamy relation in the even case ($m = 0$) but ceases to be monogamous in the odd case ($m = 1$) when the overlap p is greater than 0.8. This indicates also that the monogamy relation is violated in three qubit states of W type obtained in the limiting case $p \longrightarrow 1$. Similarly, we also considered the two tripartite splitting ($j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1$) and ($j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2}$) which can originate from the splitting of the spin $j = 2$. The figures 4 reveals that the monogamy relation is satisfied for even spin coherent states ($m = 0$). However, for odd spin coherent states ($m = 1$), the entanglement of formation does not follow the monogamy as p approaches the unity (see figure 4). This agrees with the result of figure 3 and confirms that in a W state comprising three qubits, the monogamy of entanglement of formation is violated.

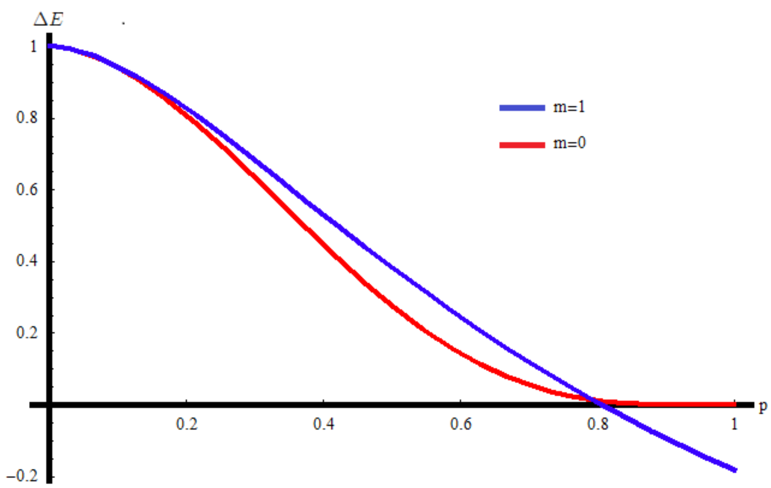

FIG. 3: The function ΔE versus the overlap p when $j_1 = j_2 = j_3 = \frac{1}{2}$ for $m = 0$ and $m = 1$.

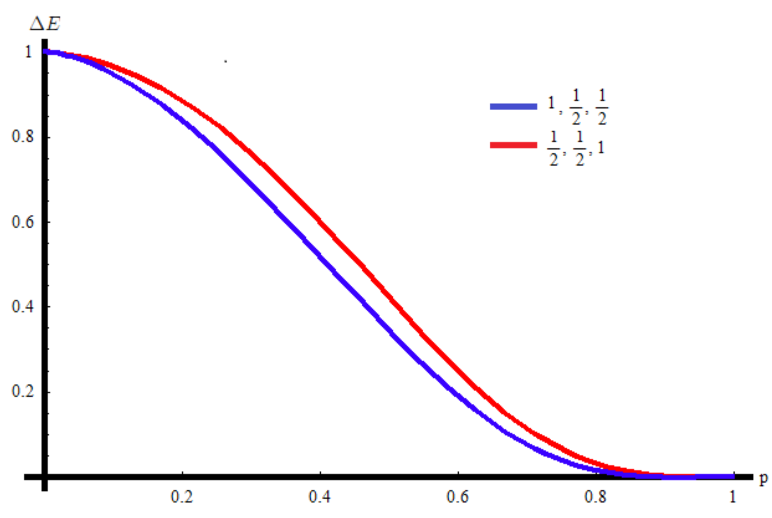

FIG. 4: The function ΔE versus the overlap p when $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1)$ and $(j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2})$ for $m = 0$.

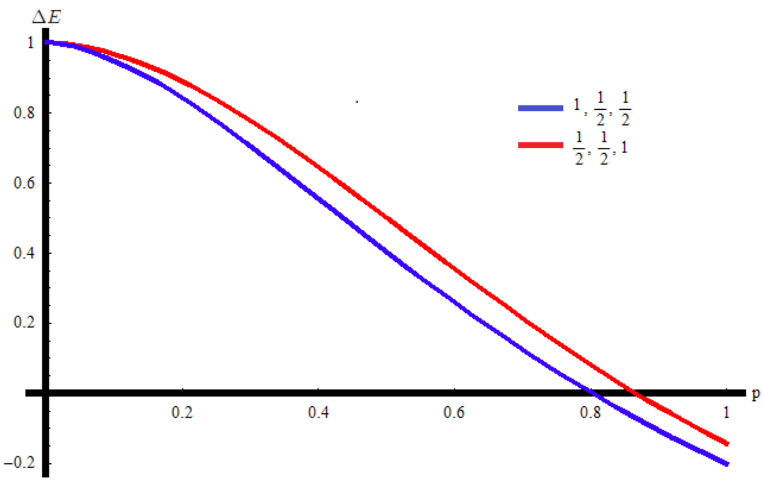

FIG. 5: The function ΔE versus the overlap p when $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1)$ and $(j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2})$ for $m = 1$.

7 Quantum discord in the three splitting scheme

7.1 Quantum discord

In the pure bipartite splitting scheme defined by (41), (42) and (43), the quantum discord and entanglement of formation as measure of bipartite quantum correlations are identical and we have

$$D(\rho_{j_1|j_2j_3}) = E(\rho_{j_1|j_2j_3}) \qquad D(\rho_{j_2|j_1j_3}) = E(\rho_{j_2|j_1j_3}) \qquad D(\rho_{j_3|j_1j_2}) = E(\rho_{j_1|j_1j_2}) \tag{57}$$

where the entanglement of formation is given by (52) modulo some obvious substitutions.
To get the explicit expressions of quantum discord in bipartite mixed states $\rho_{l_1l_2}$ of the form (50), we evaluate the mutual information entropy and the minimum of conditional entropy according to the general algorithm discussed in Section 2. We first calculate the mutual information. The non vanishing eigenvalues of the density matrix $\rho_{l_1l_2}$ are

$$\lambda_{\pm} = \frac{1}{2}\frac{(1 \pm p^{2(j-l_1-l_2)})(1 \pm p^{2(l_1+l_2)}\cos(m\pi))}{1 + p^{2j}\cos(m\pi)}, \tag{58}$$

and the joint entropy is

$$S(\rho_{l_1l_2}) = h(\lambda_+) + h(\lambda_-) = H(\lambda_+). \tag{59}$$

The eigenvalues of the marginal $\rho_{l_1} = \mathrm{Tr}_{l_2}\rho_{l_1l_2}$ are

$$\lambda_{1,\pm} = \frac{1}{2}\frac{(1 \pm p^{2(j-l_1)})(1 \pm p^{2l_1}\cos(m\pi))}{1 + p^{2j}\cos(m\pi)},$$

and the marginal entropy reads

$$S(\rho_{l_1}) = h(\lambda_{1,+}) + h(\lambda_{1,-}) = H(\lambda_{1,+}). \tag{60}$$

The eigenvalues of the marginal $\rho_{l_2} = \mathrm{Tr}_{l_1}\rho_{l_1 l_2}$ are

$$\lambda_{2,\pm} = \frac{1}{2}\frac{(1 \pm p^{2(j-l_2)})(1 \pm p^{2l_2}\cos(m\pi))}{1 + p^{2j}\cos(m\pi)},$$

and the corresponding entropy is given by

$$S(\rho_{l_2}) = h(\lambda_{2,+}) + h(\lambda_{2,-}) = H(\lambda_{2,+}). \tag{61}$$

It follows that the mutual information defined by (1) takes the form

$$I(\rho_{l_1 l_2}) = H(\lambda_{1,+}) + H(\lambda_{2,+}) - H(\lambda_+). \tag{62}$$

The second important step in deriving pairwise quantum discord requires the explicit calculation of the minimal amount of the conditional entropy (4). According the general discussion presented in the second section, it is necessary to purify the density matrix $\rho_{l_1 l_2}$ and determine the entanglement of formation of its complement. This algorithm can be achieved as follows. The matrix $\rho_{l_1 l_2}$ is a two-qubit state and subsequently decomposes as

$$\rho_{l_1 l_2} = \lambda_+ |\phi_+\rangle\langle\phi_+| + \lambda_- |\phi_-\rangle\langle\phi_-| \tag{63}$$

where the eigenvalues λ_+ and λ_- are given by (58) and the corresponding eigenstates $|\phi_+\rangle$ and $|\phi_-\rangle$ write as

$$|\phi_+\rangle = \frac{\sqrt{(1+p^{l_1})(1+p^{l_2})}}{\sqrt{2(1+p^{l_1+l_2})}}|0_{l_1}, 0_{l_2}\rangle + \frac{\sqrt{(1-p^{l_1})(1-p^{l_2})}}{\sqrt{2(1+p^{l_1+l_2})}}|1_{l_1}, 1_{l_2}\rangle \tag{64}$$

$$|\phi_-\rangle = \frac{\sqrt{(1+p^{l_1})(1-p^{l_2})}}{\sqrt{2(1(-p^{l_1+l_2})}}|0_{l_1}, 1_{l_2}\rangle + \frac{\sqrt{(1-p^{l_1})(1+p^{l_2})}}{\sqrt{2(1-p^{l_1+l_2})}}|1_{l_1}, 0_{l_2}\rangle \tag{65}$$

in the basis (49). Attaching a qubit 3 to the two-qubit system $(12) \equiv (l_1 l_2)$, we write the purification of $\rho_{l_1 l_2}$ as

$$|\phi\rangle = \sqrt{\lambda_+}|\phi_+\rangle \otimes |\mathbf{0}\rangle + \sqrt{\lambda_-}|\phi_-\rangle \otimes |\mathbf{1}\rangle \tag{66}$$

such that the whole system (123) is described by the pure density matrix $\rho_{l_1 l_2 3} = |\phi\rangle\langle\phi|$. Using the Koashi-Winter relation (9), we have

$$\widetilde{S}_{\min} = E(\rho_{23}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |\mathcal{C}(\rho_{23})|^2}) \tag{67}$$

where the concurrence of the density matrix $\rho_{23} \equiv \rho_{l_2 3}$ is

$$|\mathcal{C}(\rho_{l_2 3})|^2 = \frac{p^{4l_1}(1-p^{4l_2})(1-p^{4(j-l_1-l_2)})}{(1+p^{2j}\cos m\pi)^2}.$$

It follows that the quantum discord is then given by

$$D(\rho_{l_1 l_2}) = S(\rho_{l_1}) - S(\rho_{l_1 l_2}) + E(\rho_{l_2 3}). \tag{68}$$

Using the equations (59), (60) and (67), it rewrites explicitly as

$$
\begin{aligned}
D^{\rightarrow}(\rho_{l_1 l_2}) &= H\left(\frac{1}{2}\frac{(1+p^{2l_1})(1+p^{2(j-l_1)}\cos(m\pi))}{1+p^{2j}\cos(m\pi)}\right) \\
&- H\left(\frac{1}{2}\frac{(1+p^{2(j-l_1-l_2)})(1+p^{2(l_1+l_2)}\cos(m\pi))}{1+p^{2j}\cos(m\pi)}\right) \\
&+ H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{p^{4l_1}(1-p^{4l_2})(1-p^{4(j-l_1-l_2)})}{(1+p^{2j}\cos m\pi)^2}}\right)
\end{aligned}
\tag{69}
$$

where the pair (l_1, l_2) stands for (j_1, j_2), (j_1, j_3) and (j_2, j_3). Similarly, the measure of quantum discord obtained by measuring the second qubit $B \equiv l_2$ is

$$
\begin{aligned}
D^{\leftarrow}(\rho_{l_1 l_2}) &= H\left(\frac{1}{2}\frac{(1+p^{2l_2})(1+p^{2(j-l_2)}\cos(m\pi))}{1+p^{2j}\cos(m\pi)}\right) \\
&- H\left(\frac{1}{2}\frac{(1+p^{2(j-l_1-l_2)})(1+p^{2(l_1+l_2)}\cos(m\pi))}{1+p^{2j}\cos(m\pi)}\right) \\
&+ H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{p^{4l_2}(1-p^{4l_1})(1-p^{4(j-l_1-l_2)})}{(1+p^{2j}\cos m\pi)^2}}\right).
\end{aligned}
\tag{70}
$$

It is interesting to note that

$$
D^{\rightarrow}(\rho_{l_1 l_2}) = D^{\leftarrow}(\rho_{l_2 l_1}). \tag{71}
$$

It is clear that for $l_1 = l_2$, the quantum discord is symmetric, i.e. $D^{\rightarrow}(\rho_{ll}) = D^{\leftarrow}(\rho_{ll})$. Using the equation (69), one obtains the following conservation relations

$$
\begin{aligned}
D^{\rightarrow}(\rho_{j_1 j_2}) + D^{\rightarrow}(\rho_{j_3 j_2}) &= E_{j_2 j_3} + E_{j_2 j_1}, \\
D^{\rightarrow}(\rho_{j_2 j_1}) + D^{\rightarrow}(\rho_{j_3 j_1}) &= E_{j_1 j_3} + E_{j_1 j_2}, \\
D^{\rightarrow}(\rho_{j_1 j_3}) + D^{\rightarrow}(\rho_{j_2 j_3}) &= E_{j_3 j_2} + E_{j_3 j_1}.
\end{aligned}
\tag{72}
$$

Similar conservations relations hold for the measures of quantum discord given by (70). They can be easily derived from the relation (71). Using the conservation relations (72), we have

$$
D^{\rightarrow}(\rho_{j_1 j_2}) + D^{\rightarrow}(\rho_{j_2 j_3}) + D^{\rightarrow}(\rho_{j_3 j_1}) = E_{j_1 j_2} + E_{j_1 j_3} + E_{j_2 j_3}.
$$

This reflects that the sum of pairwise quantum discord for all bipartite mixed states coincides with the sum of entanglement of formation. It must be noticed that the conservation relations of type (72) involving entanglement of formation and quantum discord were first derived in [38].

7.2 Multipartite quantum correlations

Based on the asymmetric definition of quantum discord, two interesting quantities were defined by Fanchini et al [52]. In our context, they write

$$
\Delta^{+}_{l_1|l_2} = \frac{1}{2}\big(D^{\rightarrow}(\rho_{l_1 l_2}) + D^{\rightarrow}(\rho_{l_2 l_1})\big), \tag{73}
$$

and

$$\Delta^{-}_{l_1|l_2} = \frac{1}{2}\big(D^{\rightarrow}(\rho_{l_1l_2}) - D^{\rightarrow}(\rho_{l_2l_1})\big). \tag{74}$$

The sum $\Delta^{+}_{l_1|l_2}$ is the average of locally inaccessible information when the measurements are performed on the subsystems l_1 and l_2. It quantifies the disturbance caused by any local measurement. The difference $\Delta^{-}_{l_1|l_2}$ is the balance of locally inaccessible information and quantifies the asymmetry between the subsystems in responding to the measurement disturbance. Using the equation (69), it is easy to verify that the average and the balance of quantum discord satisfy the following identities

$$\Delta^{+}_{j_1|j_2} + \Delta^{+}_{j_1|j_3} + \Delta^{+}_{j_2|j_3} = E_{j_1j_2} + E_{j_1j_3} + E_{j_2j_3}, \tag{75}$$

and

$$\Delta^{-}_{j_1|j_2} + \Delta^{-}_{j_1|j_3} + \Delta^{-}_{j_2|j_3} = 0. \tag{76}$$

Using the main definition (14), it is interesting to note that the total amount of quantum discord present in the state (40) can be simply written in terms of the average of locally inaccessible information (73). Indeed, we have

$$D(j_1, j_2, j_3) = \frac{1}{6}\left(\Delta^{+}_{j_1|j_2} + \Delta^{+}_{j_1|j_3} + \Delta^{+}_{j_2|j_3} + \Delta^{+}_{j_1|(j_2j_3)} + \Delta^{+}_{j_2|(j_1j_3)} + \Delta^{+}_{j_3|(j_1j_2)}\right) \tag{77}$$

where the quantity $\Delta^{+}_{k_1|(k_2k_3)}$ coincides with the entanglement of formation $E(\rho_{k_1|k_2k_3})$ given by (52). Furthermore, using the conservation relation (75), one gets

$$D(j_1, j_2, j_3) \quad = \quad E(j_1, j_2, j_3) \tag{78}$$

where $E(j_1, j_2, j_3)$ is given by (56). This result coincides with one obtained in [38]. It reflects that the sum of quantum discord present in all possible bi-partitions is exactly the total amount of bipartite entanglement of formation in the entire system.

Since for a spin-j coherent state there are different tripartite splitting possibilities denoted here by (j_1, j_2, j_3) such that $j_1 + j_2 + j_3 = j$, it is seems natural to compare the total amount of multipartite correlations in each splitting scheme. As illustration, we consider the situation where $j = 3$. The tripartite quantum discord $D(j_1, j_2, j_3)$ (78) is totally symmetric in j_1, j_2 and j_3. Thus, for $j = 3$, three inequivalent splitting schemes are of special interest. They correspond to $(j_1 = 1, j_2 = 1, j_3 = 1)$, $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 2)$ and $(j_1 = \frac{1}{2}, j_2 = 1, j_3 = \frac{3}{2})$. In figures 6 and 7, we plot the quantity $D(j_1, j_2, j_3)$ as function of the overlap p for each case.

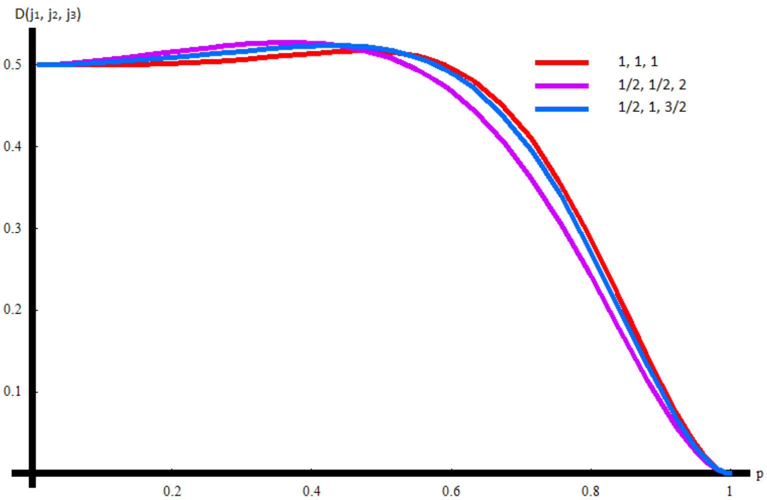

FIG. 6: The multipartite quantum correlations for $j = 3$ versus the overlap p for $m = 0$.

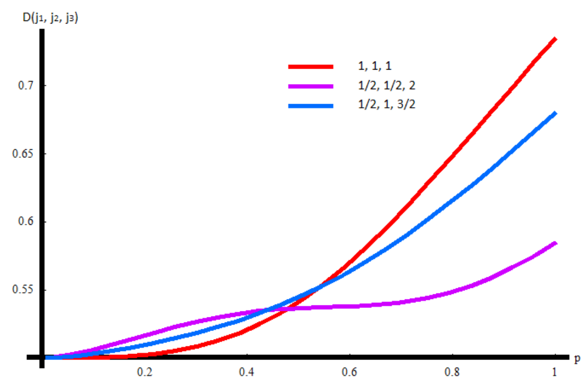

FIG. 7: The multipartite quantum correlations for $j = 3$ versus the overlap p for $m = 1$.

From figures 6 and 7, one can see that the tripartite quantum discord $D(j_1 = 1, j_2 = 1, j_3 = 1)$, $D(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 2)$ and $D(j_1 = \frac{1}{2}, j_2 = 1, j_3 = \frac{3}{2})$ are all equals for $p \simeq 0.5$. Note also that for $p \leq 0.5$, the sum of all pairwise quantum discord obtained in the spitting scheme $(j = 3) \longrightarrow (j_1 = 1, j_2 = 1, j_3 = 1)$ is minimal in comparison with the two others. This behavior changes when $p \geq 0.5$ and the quantity $D(j_1 = 1, j_2 = 1, j_3 = 1)$ becomes maximal. For even spin coherent states $(m = 0)$, the measure of tripartite quantum correlations vanishes when $p \longrightarrow 1$ as expected (see equations (56) and (78)).

7.3 Monogamy of quantum discord

In the pure tripartite state (40), the quantum discord satisfy the monogamy relation when the following condition

$$D^{\rightarrow}(\rho_{j_1 j_2}) + D^{\rightarrow}(\rho_{j_1 j_3}) \leq D^{\rightarrow}(\rho_{j_1|j_2 j_3})$$

is satisfied. As for entanglement of formation, we shall focus on some special cases to determine the positivity of the function

$$\Delta D = D^{\rightarrow}(\rho_{j_1|j_2 j_3}) - D^{\rightarrow}(\rho_{j_1 j_2}) - D^{\rightarrow}(\rho_{j_1 j_3})$$

when the overlap vary from 0 to 1. We first consider the situation where ($j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = \frac{1}{2}$). The function ΔD is plotted in figure 8. In this case the quantum discord is monogamous for even spin coherent state. However, for odd spin coherent state, the monogamy relation is satisfied only when $p \leq 0.8$. We also consider the situations where ($j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2}$), ($j_1 = \frac{1}{2}, j_2 = 1, j_3 = \frac{1}{2}$) and ($j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1$) associated to the spin $j = 2$. The behavior of the function ΔD for even coherent states ($m = 0$) is reported in the figure 9. Clearly, the monogamy relation is satisfied. The figure 10, representing the function ΔD for odd case ($m = 1$), reveals that the quantum discord ceases to be monogamous for p approaching the unity. Remark that in the figures 9 and 10, we have $\Delta D(\frac{1}{2}, 1, \frac{1}{2}) = \Delta D(\frac{1}{2}, \frac{1}{2}, 1)$ as expected. It is interesting to note that the behavior of ΔD versus p is identical to the ones obtained for ΔE in the previous section (figures 3, 4 and 5). This is essentially due to the conservation relations between quantum discord and entanglement of formation (72) [38]. Finally, it is interesting to note that the odd tripartite coherent states ($m = 1$) interpolate continuously between the three-qubit Greenberger-Horne-Zeilinger (GHZ_3) states when $p \to 0$ and W_3 states for $p \to 1$. It follows from figure 10 that the GHZ_3 states follow monogamy and W_3 states do not.

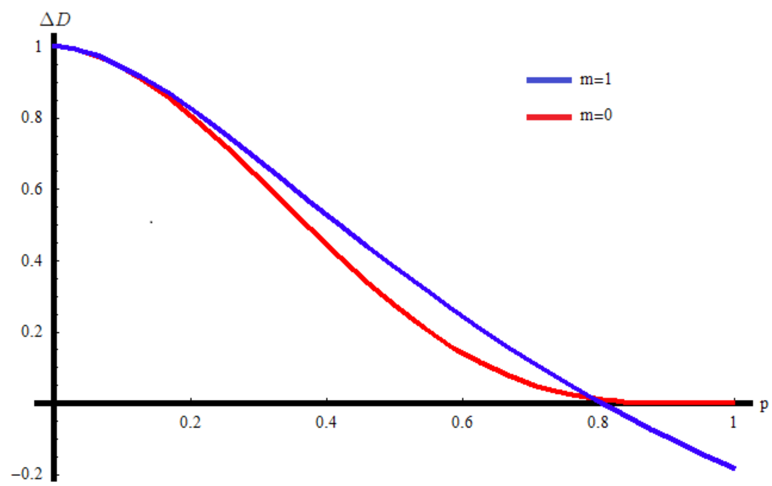

FIG. 8: The function ΔD versus the overlap p when $j_1 = j_2 = j_3 = \frac{1}{2}$ for $m = 0$ and $m = 1$.

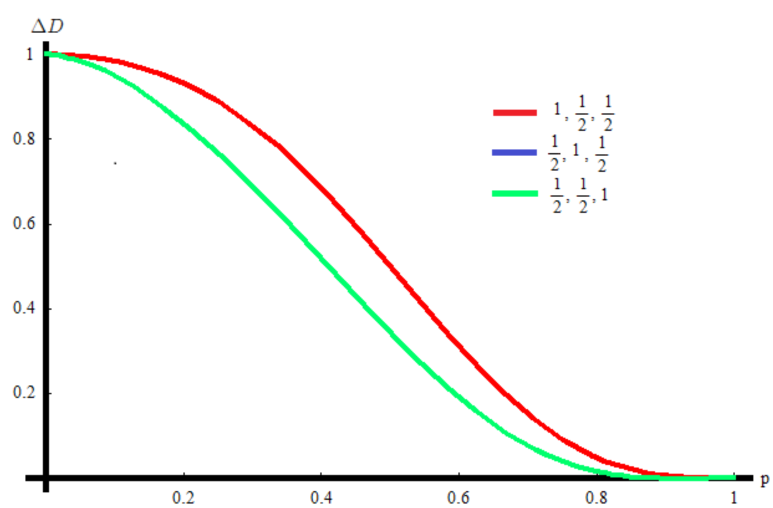

FIG. 9: The function ΔD versus the overlap p when ($j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2}$) and

$(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1)$ for $m = 0$.

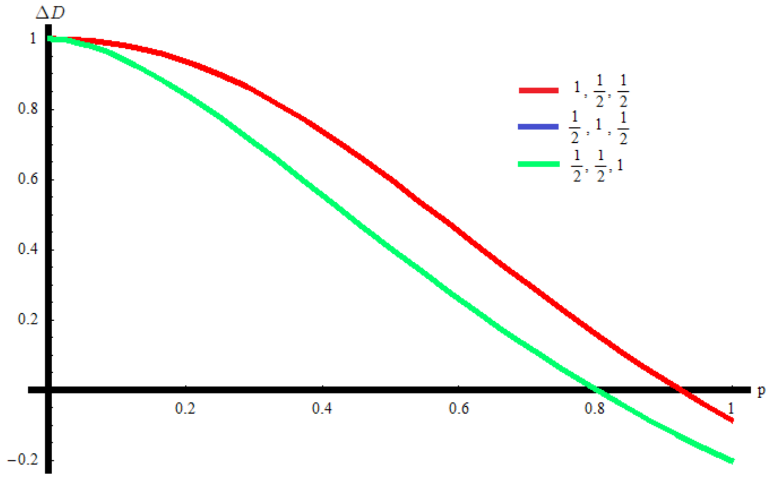

FIG. 10: The function ΔD versus the overlap p when $(j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2})$ and $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1)$ for $m = 1$.

8 Concluding remarks

The main motivation in investigating the multipartite quantum correlations in even and odd coherent states is the decomposition (or factorization) property given by (23). In this way, a single j-spin coherent state is viewed as comprising two, three or in general $2j$ qubits. Moreover, this decomposition property allows us to investigate the pairwise quantum correlations in a in a single spin coherent state. In this paper, we mainly focused on bipartite and tripartite decomposition. For each case, the spin coherent states were mapped to two or three qubits system. We have considered the multipartite quantum correlation in even and odd spin coherent states measured by entanglement of formation and quantum discord. We defined the total amount of quantum correlation in spin coherent states, viewed as multi-components system, as the sum of all pairwise quantum correlations. We explicitly derived the expressions of multipartite entanglement of formation and quantum discord for even and odd spin coherent states. The sum of all possible pairwise entanglement of formation in an even or odd spin coherent, viewed as a pure tripartite state, is explicitly derived and it coincides with sum of pairwise quantum discord of all possible bi-partitions as it has been shown in [38]. This peculiar result originates from the conservation relation between the entanglement of formation and quantum discord given by (72). We also examined the monogamy relation of entanglement of formation and quantum discord. Remarkably, in the simplest cases that we considered, these two measures are monogamous for even spin coherent contrarily to odd case where the monogamy relation is violated for states involving an overlap p approaching the unity. In particular, we have shown that the entanglement of formation and quantum discord follow the monogamy relation in the three qubit Greenberger-Horne-Zeilinger states contrarily to the three qubit states of W type. As prolongation of the present work, it will be an important issue to extend the present approach to others coherent and squeezed states. Further thought in this direction might be worthwhile in investigating genuine multipartite quantum correlations. Finally, it is interesting to examine the relation between the spin coherent states factorization (23) and the tensor product decomposition of two fermions developed in [53].

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