Quantum optomechanical piston engines powered by heat

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We study two different models of optomechanical systems where a temperature gradient between two radiation baths is exploited for inducing self-sustained coherent oscillations of a mechanical resonator. Viewed from a thermodynamic perspective, such systems represent quantum instances of self-contained thermal machines converting heat into a periodic mechanical motion and thus they can be interpreted as nano-scale analogues of macroscopic piston engines. Our models are potentially suitable for testing fundamental aspects of quantum thermodynamics in the laboratory and for applications in energy efficient nanotechnology.

The research on quantum thermodynamics received large attention since the beginning of quantum physics. Its main task is understanding to what extent the laws of thermodynamics are valid in the quantum regime [1–10]. In particular one of the main questions which are currently considered is how much can thermal machines (heat engines and refrigerators) be miniaturized while retaining their essential feature of producing work or extracting heat [5, 8, 11–14].

In this paper we propose simple models of microscopic piston engines based on quantum optomechanical systems [15, 16], *i.e.* devices composed of micro/nanoscale mechanical resonators coupled to optical or microwave modes. In the last few years exceptional levels of quantum control over optomechanical systems have been reached. For example important milestones like ground state cooling of a mechanical resonator [17–19], squeezing [20] and optomechanical entanglement [21] have been recently experimentally achieved. These facts suggest that the research level on optomechanics is sufficiently advanced to allow implementations of quantum thermodynamics ideas with near-future technology.

Our specific contribution is the proposal of two optomechanical setups, that we call *single cavity engine* and *cascade engine*, in which a temperature gradient between two thermal baths is exploited for inducing self-sustained oscillations of a mechanical resonator. The emergence of persistent mechanical oscillations in a system which is subject to friction and dissipation can be interpreted as a continuous production of thermodynamic work. Alternatively the oscillating degree of freedom can be used as a resource for producing work on some additional external system, in analogy with common macroscopic piston engines. In our analysis we first give a proof-of-principle demonstration of the possibility of inducing mechanical self-sustained oscillations from thermal noise and without external forces. Then we estimate the power of the engines from a thermodynamic perspective and we also underline the differences between classical and quantum optomechanical motors. We stress that, since our devices are in contact with independent baths at different temperatures, the spontaneous emergence of persistent self-sustained oscillations is not an example of any paradoxical perpetual motion and does not violate the second law of thermodynamics.

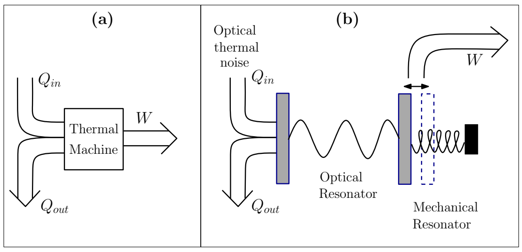

FIG. 1: (a) Scheme of a general heat engine: input heat Q_{in} is absorbed, output heat Q_{out} is dissipated and output work W is produced. (b) Model of an equivalent quantum optomechanical thermal motor. Heat is absorbed from a hot optical/microwave thermal bath. This energy is partially used to excite the coherent motion of a mechanical resonator (work) and the rest is dissipated into a cold optical/microwave bath.

In the last years other models of optomechanical engines have been proposed, in which the systems are driven by periodic coherent lasers and undergo thermodynamic cycles [22–24]. Instead, the key feature characterizing our optomechanical engines is that they are self-contained since they are driven by pure heat and not by external forces. In this sense our approach is similar to the analysis of the finite dimensional thermal machines introduced in [12, 25], to the "cooling by heating" setup proposed in [26] and to the concept of Brownian motors reviewed in [27, 28]. In the field of optomechanics our work settles in the framework of opto-mechanical lasing [29–33] where, however, the effect is usually induced by coherent external drivings. Our results are also related to the experiment reported in Ref. [34] where classical colored noise has been applied to excite oscillations of a nonlinear mechanical system in the classical regime. The thermodynamic interpretation of a lasing (optical or microwave) system as a quantum heat engine can be traced back to the seminal work by Scovil and Schulz-DuBois [35] and has been studied more recently in the context of hybrid (continous-descrete) systems [8, 36–38].

The possibility of realizing heat powered microscopic piston engines could find interesting technological applications in the current research on energetically efficient nano-scale devices [38–40]. At the same time, in the same spirit of other previous proposals [22, 24, 26, 41], the systems introduced in this work could be used as convenient toy models for testing fundamental aspects of quantum thermodynamics in the laboratory.

Single cavity engine – The first system that we consider involves a mechanical resonator of frequency ω_c coupled by radiation pressure to two radiation modes of frequency ω_a and ω_b respectively. The corresponding Hamiltonian is

$$\begin{aligned} H = \;& \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\omega_c c^\dagger c \\ & -\hbar g(a+b)^\dagger(a+b)(c+c^\dagger), \end{aligned} \tag{1}$$

where a, b, c are the bosonic annihilation operators of the three modes and g is the optomechanical coupling constant. The last term in Eq. (1) is proportional to the position of the mechanical resonator and to the intensity of the cavity field. For example this Hamiltonian could describe the radiation pressure of a cavity field on a moving mirror [15, 16], but the same model well represents extremely different systems like toroidal micro-cavities [42], opto-mechanical crystals [17], cold atoms [43], *etc.*. This model applies as well to electro-mechanical systems where the radiation modes a and b have frequencies in the microwave range [18, 21].

The three modes are put in contact with three independent environments, which can possess different temperatures. The corresponding dynamics of the open system, in the weak coupling limit, is well described by the following master equation [44]:

$$\begin{aligned} \dot\rho = -\frac{i}{\hbar}[H,\rho] \;& +\kappa_a(N_a+1)D_a(\rho) + \kappa_a N_a D_{a^\dagger}(\rho) \\ & +\kappa_b(N_b+1)D_b(\rho) + \kappa_b N_b D_{b^\dagger}(\rho) \\ & +\kappa_c(N_c+1)D_c(\rho) + \kappa_c N_c D_{c^\dagger}(\rho), \end{aligned} \tag{2}$$

where the $D_x(\cdot)$ is the Lindblad dissipator $D_x(\rho) = x\rho x^\dagger - \frac{1}{2}\{x^\dagger x, \rho\}$ associated with the modes $x = a, b, c$, κ_x is the decay rate, and N_x depends on the temperature T_x of the respective environment according to the Bose-Einstein statistics $N_x = [e^{\frac{\hbar\omega_x}{k_B T_x}} - 1]^{-1}$. The quantity N_x is a monotonously increasing function of T_x and in what follows we parametrize the temperature in terms of N_x.

As we are going to show, if the resonance $\omega_b - \omega_a = \omega_c$ is satisfied and if the thermal noise parameter N_b is large enough, then it is possible to excite mechanical self-sustained oscillations of the mode c. Before presenting the results in details, let us first introduce also the second model of optomechanical engine.

Cascade engine – In the previous model (single cavity engine) two optical modes were supported by the same optomechanical cavity. For technical reasons it may be more practical to realize a cascade engine where the mode b is associated with an independent optical filter (*e.g.* a Fabry-Pérot resonator) whose output is fed into a standard optomechanical system based on a single optical mode. It turns out that this setting provides results which are qualitatively equivalent to the single cavity setup and, at the same time, it could be experimentally easier to realize. For example the required tuning of the resonance condition $\omega_b - \omega_a = \omega_c$ should be much simpler if the two modes a and b are supported by two separeted devices.

The Hamiltonians associated with the first and second cavities are respectively:

$$H_1 = \hbar\omega_b b^\dagger b, \tag{3}$$

$$H_2 = \hbar\omega_a a^\dagger a + \hbar\omega_c c^\dagger c - \hbar g a^\dagger a(c+c^\dagger). \tag{4}$$

In addition to the dissipative channels that we introduced in the single cavity engine, here we also have to take into account that the output of the first cavity is fed into the second one. The corresponding master equation can be derived using the quantum optics framework of *cascaded quantum systems* [44, 45], obtaining

$$\begin{aligned} \dot\rho = \;& -\frac{i}{\hbar}[H_1 + H_2, \rho] \\ & +\kappa_a(N_a+1)D_a(\rho) + \kappa_a N_a D_{a^\dagger}(\rho) \\ & +\kappa_c(N_c+1)D_c(\rho) + \kappa_c N_c D_{c^\dagger}(\rho) \\ & +\gamma_1 D_b(\rho) + \gamma_2 D_a(\rho) - \sqrt{\gamma_1\gamma_2}([a^\dagger, b\rho] + [\rho b^\dagger, a]) \\ & +\frac{N_b}{2}\left[[\sqrt{\gamma_1}b + \sqrt{\gamma_2}a, \rho], \sqrt{\gamma_1}b^\dagger + \sqrt{\gamma_2}a^\dagger\right] \\ & +\frac{N_b}{2}\left[[\sqrt{\gamma_1}b^\dagger + \sqrt{\gamma_2}a^\dagger, \rho], \sqrt{\gamma_1}b + \sqrt{\gamma_2}a\right]. \end{aligned} \tag{5}$$

For convenience of the reader, we comment that Eq. (5) is basically equal to Eq. (12.1.16) of Ref. [44]. The first three lines of Eq. (5) are analogous to the previous model. The last three lines instead describe a cascade setup in which the light exiting the first cavity with a rate γ_1 is fed into the second cavity with a rate γ_2. In the following we will set for simplicity $\gamma_1 = \gamma_2 = \kappa_a$, considering a scenario in which the filter and the optomechanical system consist on two Fabry-Pérot cavities with equal finesse.

Classical engine – In order to investigate the difference between classical and quantum thermal machines we will also compare the single-cavity engine with its own classical version. The classical model is obtained interpreting the Hamiltonian of Eq. (1) as being described by classical position and momentum quadratures. The effect of thermal fluctuations is then included by simulating stochastic noise in the equations of motion. More details are given in the Supplemental Material.

Self-sustained oscillations powered by heat – The possibility of inducing coherent self-sustained oscillations in optomechanical systems has been theoretically [29–32] and experimentally demonstrated [33, 34], and it is nowadays a well established technique. However in our optomechanical engines there is not a driving term in the Hamiltonian and the only source of energy is provided by the incoherent absorption of heat. It is therefore not guaranteed that coherent oscillations can emerge in our setups and the main task of this work is to give a proof of principle demonstration that this effect is actually possible. In a second step we will study some thermodynamic aspects of the engines and compare the classical and quantum versions of the motors.

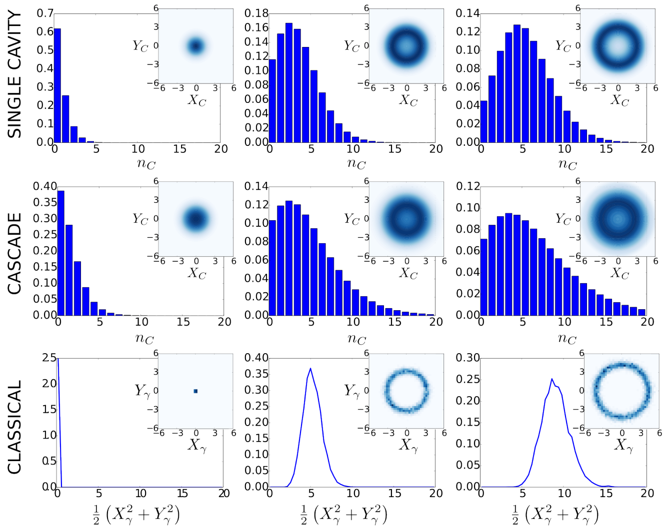

FIG. 2: From left to right, phonon number distributions and Wigner functions (insets) for different values of the thermal noise parameter $N_b = 0.17, 0.33, 0.5$ ($N_b = 0$ trivially gives the vacuum state). The first line refers to the single cavity engine with parameters $N_a = N_c = 0$, $\omega_b - \omega_a = \omega_c = 1$, $\kappa_a = \kappa_b = 0.2$, $\kappa_c = 0.005$ and $g = 0.06$. The second line refers to the cascade engine with $\kappa_a = \gamma_1 = \gamma_2 = 0.15$, $\kappa_c = 0.003$ and $g = 0.1$. The third line instead represents the classical version of the single cavity engine (see Supplemental Material).

In the standard theory of optomechanical limit cycles, the driving laser is chosen with a frequency larger than the cavity resonance and such that the detuning matches the frequency of the mechanical resonator. From this fact we learn that, if we wish to have self-sustained oscillations in our engines, energy should be put in the radiation mode of larger frequency while the other mode should be as pure as possible in order to absorb and dissipate the photons scattered by the mechanical resonator. The optimal choice of temperatures is therefore $N_a = N_c = 0$ and $N_b > 0$. For the other system parameters we consider typical values which are known to allow limit cycles in the presence of a coherent laser [31, 32] and, as we are going to show, these values remain suitable also in our dissipative setups. The specific parameters are reported in the caption of Fig. 2 and are consistent with the recent experimental advances in strongly coupled optomechanical systems [17, 18, 21, 43]. We then vary the temperature of the bath of the mode b (*i.e.* we increase N_b) and we numerically solve the steady state condition $\dot{\rho} = 0$ associated to the master equation of the single cavity engine (2) and of the cascade engine (5). The steady state is found exactly (without rotating wave approximations) in a truncated Fock space of up to 3 photons for the modes a and b and 20 phonons for the mode c.

The numerics has been performed using the open-source toolbox QuTiP2 [46], and the results are shown in Fig. 2. From the sequence of Wigner functions evaluated for increasing values of N_b it is clear that the mechanical resonator is initially heated up in a thermal state and above a given threshold it develops a limit-cycle with the characteristic ring shape in phase space. The same effect is evident also in the probability distribution of the number of phonons in the system (diagonal elements of ρ in the Fock basis), where the transition is from a Gibbs distribution to a Poissonian one typical of a coherent state. We can thus claim that, in this regime our optomechanical engines are effectively behaving as quantum piston engines converting heat into coherent mechanical oscillations.

A remark should be made about the notion of "coherent" oscillations. From the shape of the Wigner function one can see that the steady state of the mechanical oscillator is actually phase randomized and the density matrix is essentially diagonal in the Fock basis. The randomization of the phase is the unavoidable consequence of the rotation symmetry of the system and corresponds exactly to the same feature possessed by the steady states of standard optical lasers. The notion of coherence which then applies in our case is the standard criterion used in quantum optics for distinguishing between thermal and coherent radiation, namely the equal-time normalized second-order coherence function [44]: $g_2 = \langle c^\dagger c^\dagger c c\rangle / \langle c^\dagger c\rangle^2$. Basically g_2 measures how likely it is to consecutively detect two phonons at a given instant of time. For thermal states one has $g_2 = 2$ (bounching statistics), while for coherent states $g_2 = 1$ (Poissonian statistics). In general if the quantity g_2 decreases from the thermal threshold of 2 towards lower values, then this is a hint that the field is developing some level of coherence and therefore that a lasing effect is happening in the system. In Fig. 3.a the quantity g_2 is plotted for different values of N_b, quantitatively showing the transition of the quantum state of the mechanical resonator from an incoherent to a coherent one.

Power of the quantum engine – As noticed in [35] and further investigated in [36–38, 47], the energy of a coherent field can be interpreted as thermodynamic work and the lasing device as a heat engine. However a quantitative and rigorous analysis of the work produced by the engine is a non-trivial fundamental problem. The task of quantifying the maximum work (or power) extractable form a quantum system is still subject to a significant research effort [1–5, 7–9, 48]. The dynamics of our motors is non-cyclic and open (non-unitary) and it is not obvious what is the amount of work produced by a mechanical resonator which is continuously sustained in a non equilibrium steady state. We then adopt a pragmatic approach and try to give an estimate of the power by indirectly considering the energy dissipated by the mechanical resonator into its environment. This can be explicitly computed [36] giving :

$$\begin{aligned} P &= -\mathrm{Tr}\{\hbar\omega_c c^\dagger c \left[\kappa_c(N_c+1)D_c(\rho) + \kappa_c N_c D_c^\dagger(\rho)\right]\} \\ &= \hbar\omega\kappa_c(\langle c^\dagger c\rangle - N_c). \end{aligned} \quad (6)$$

The dissipated power, has some obvious problems since it does not distinguishes between useless energy (heat)

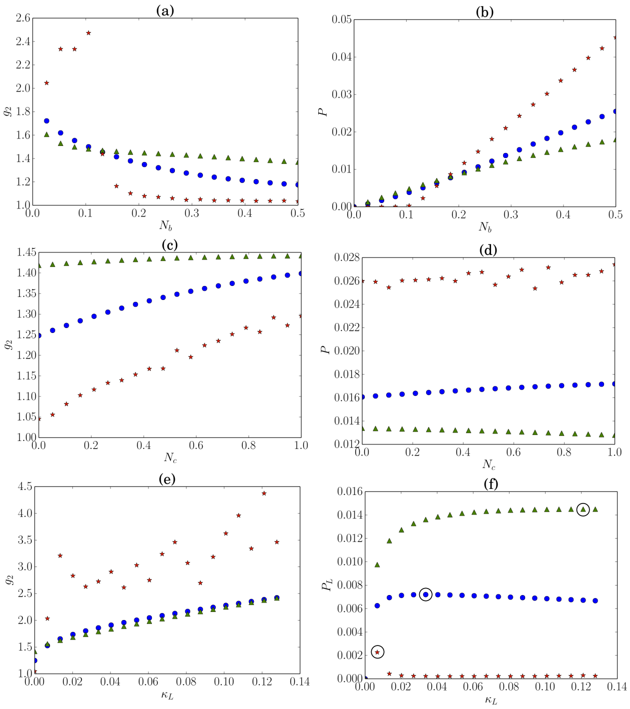

FIG. 3: (a) Equal-time second order coherence function g_2 and (b) dissipated power P (in units of $\hbar\omega_c$ per second) with respect to the thermal noise parameter N_b. The other parameters are the same as those used in Fig. 2. In (c) and (d), the same quantities are plotted with respect to the mechanical bath mean phonon number N_c, for $N_b = 0.33$. In (e) and (f), g_2 and the external power P_L (in units of $\hbar\omega_c$ per second) are evaluated with respect to the load damping rate κ_L. The maxima of P_L are highlighted by black circles. For all plots the legend is: single cavity engine (blue circles), cascade engine (green triangles), classical engine (red stars).

resulting from Brownian fluctuations and useful energy (work) (a similar issue has been discussed in Ref. [37]). Nonetheless at least when we are in the lasing regime, *i.e.* when the motion of the mechanical mode is coherent, the dynamics is similar to a classical harmonic oscillator rotating in a deterministic circular phase-space orbit. For a classical oscillator it is clear that the energy of the limit cycle can be easily converted into useful work. Then we can argue that, if a system is in a coherent limit cycle, the dissipated power is a reasonable figure of merit of the work extractable form the system. It would be an interesting problem to understand how a quantum mechanical limit cycle can be rectified in order to lift a “weight” (excite a work medium) in the quantum regime, in the same spirit of [7, 48, 49]. This and other quantitative thermodynamic analysis are, however, outside the “proof of principle” approach of this work and will be investigated elsewhere.

In Fig. 3.b the dissipated power is shown as a function of N_b for the single-cavity, the cascade, and the classical engines. We observe that the two quantum models are qualitatively equivalent even if the single cavity engine performs quantitatively better for fixed values of the parameters. For large N_b the classical engine seems more powerful than the quantum counterpart. Interestingly however, due to a sharper lasing transition, the classical engine is not able to excite the mechanical resonator for small values of N_b. These discrepancies could be associated to the presence of quantum fluctuations in the dynamics of the quantum engine: these fluctuations are deleterious for large N_b but, by smoothing the lasing transition, they became advantageous for small values of N_b. Finally in Fig. 3(c) and 3(d) we report the quantities g_2 and P for non-zero values of temperature of the mechanical bath. The dissipated power is slightly modified for larger values of N_c, while the quantity g_2 is instead increased towards thermal-like statistics. A possible interpretation of this fact is that, for $N_c \neq 0$, a larger fraction of power is dissipated in the form of heat rather than work.

Maximum external power – The quantity P is the intrinsic power needed by the mechanical resonator to continuously sustain its coherent oscillations. However, in analogy with macroscopic engines, one could imagine to use these oscillations for doing work on an external load. This load, will act as an additional friction force damping the mechanical mode. Then we can model a generic load substituting $\kappa_c \to \kappa_c + \kappa_L$ in Eq.s (2) and (5), where κ_L is the damping constant due to the load. The power P_L externally dissipated by the load will be given by the same expression of Eq. (6) but with the substitution $\kappa_c \to \kappa_L$. It is clear that the external power is zero in the two limits $\kappa_L = 0$ and $\kappa_L \to \infty$, then there must exist an optimal load κ_L maximizing P_L. This behavior is confirmed by our numerical results presented in Fig. 3.f. We also computed the g_2 function with respect to the load parameter κ_L (Fig. 3.e) and, as expected, the load tends to destroy the coherence of the mechanical mode.

Discussion – We proposed two different models of optomechanical engines based on a single cavity and a cascade setup respectively. In both cases we have shown that random thermal fluctuations of optical or microwave fields can be exploited for inducing self-sustained coherent oscillations of a mechanical resonator. In this regime, our systems behave as nano-scale analogues of macroscopic piston engines driven by thermal energy. We estimated the dissipated power of the optomechanical motors and highlighted the differences between quantum and classical engines.

We believe that our analysis, together with other recent ideas [22, 24, 26, 41], could pave the way for the development of fundamental experiments on quantum thermodynamics based on optomechanical systems. At the same time the paradigm of piston engines presented in this work could find practical technological applications in the fabrication of micro-mechanical motors [27, 28] and energy efficient nano-scale devices [39, 40].

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Supplemental Material: Quantum optomechanical piston engines powered by heat

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In this supplemental material we describe the classical analogue of the quantum single cavity engine presented in the main text.

Classical single cavity engine – The single cavity engine involves a mechanical resonator of frequency ω_c coupled by radiation pressure to two radiation modes of frequency ω_a and ω_b respectively. For the quantum case we have the Hamiltonian (1) (see main text) which gives the following Heisenberg equations for the bosonic annihilation operators a, b, c of the three modes:

$$\begin{aligned} \dot{a} &= -i\Delta a + ig(a+b)(c^\dagger + c), \\ \dot{b} &= +ig(a+b)(c^\dagger + c), \\ \dot{c} &= -i\omega_c c + ig(a+b)^\dagger(a+b). \end{aligned} \tag{S1}$$

Please note that we expressed the radiation operators in a frame rotating with frequency ω_b. We also defined the detuning $\Delta = \omega_a - \omega_b$ and set it to be $\Delta = -\omega_c$.

We obtain the classical counterpart of Eq.s (S1) by demoting the operators a, b, c to classical dynamical complex amplitudes α, β, γ (essentially reversing the standard quantization procedure):

$$\begin{aligned} \dot{\alpha} &= -i\Delta\alpha + ig(\alpha+\beta)(\gamma^* + \gamma), \\ \dot{\beta} &= +ig(\alpha+\beta)(\gamma^* + \gamma), \\ \dot{\gamma} &= -i\omega_c\gamma + ig|\alpha+\beta|^2. \end{aligned} \tag{S2}$$

The three oscillators are put in contact with three independent environments, which can possess different temperatures. We then add friction terms and classical Brownian noises to Eq.s (S2), turning them into classical Langevin equations:

$$\begin{aligned} \dot{\alpha} &= -i\Delta\alpha + ig(\alpha+\beta)(\gamma^* + \gamma) - \frac{\kappa_a}{2}\alpha + \xi^a, \\ \dot{\beta} &= +ig(\alpha+\beta)(\gamma^* + \gamma) - \frac{\kappa_b}{2}\beta + \xi^b, \\ \dot{\gamma} &= -i\omega_c\gamma + ig|\alpha+\beta|^2 - \frac{\kappa_c}{2}\gamma + \xi^c. \end{aligned} \tag{S3}$$

In the above expressions κ_a, κ_b and κ_c are the dissipation rates (we keep the same values as in the quantum engine), while $\xi^a = (\xi^a_x + i\xi^a_y)/\sqrt{2}$, $\xi^b = (\xi^b_x + i\xi^b_y)/\sqrt{2}$ and $\xi^c = (\xi^c_x + i\xi^c_y)/\sqrt{2}$ are independent complex zero-mean Gaussian white noises with correlations $\langle \xi^\nu_x(t)\xi^\nu_x(t')\rangle = \langle \xi^\nu_y(t)\xi^\nu_y(t')\rangle = \kappa_\nu N_\nu \delta(t-t')$ $(\nu = a, b, c)$.

We stress that this model should not be interpreted as a semi-classical approximation of the quantum system. Instead it represents a purely classical description of the optomechanical system which, in principle, one could derive directly form classical electromagnetism. Indeed our aim is not to approximate the quantum model with the classical one but, on the contrary, to understand the differences between the quantum and the classical optomechanical engines.

Simulation – The differential equations (S3) make sense only with respect to stochastic integration [S1]. In simple words, for simulating the dynamics, we must take finite increments over a small time step dt:

$$\begin{aligned} d\alpha &= \Big[-i\Delta\alpha + ig(\alpha+\beta)(\gamma^*+\gamma) - \frac{\kappa_a}{2}\alpha\Big]dt + \frac{dW^a_x + idW^a_y}{\sqrt{2}}, \\ d\beta &= \Big[+ig(\alpha+\beta)(\gamma^*+\gamma) - \frac{\kappa_b}{2}\beta\Big]dt + \frac{dW^b_x + idW^b_y}{\sqrt{2}}, \\ d\gamma &= \Big[-i\omega_c\gamma + ig|\alpha+\beta|^2 - \frac{\kappa_c}{2}\gamma\Big]dt + \frac{dW^c_x + idW^c_y}{\sqrt{2}}. \end{aligned} \tag{S4}$$

where now dW^ν_x and dW^ν_y $(\nu = a, b, c)$ are independent random increments sampled from Gaussian distributions with zero mean and variances equal to $\sqrt{\kappa_\nu N_\nu dt}$. It is convenient to recast Eq.s (S4) in terms of the real adimensional position and momentum variables X_ν, Y_ν $(\nu = \alpha, \beta, \gamma)$, defined such that $\alpha = (X_\alpha + iY_\alpha)/\sqrt{2}$, $\beta = (X_\beta + iY_\beta)/\sqrt{2}$ and $\gamma = (X_\gamma + iY_\gamma)/\sqrt{2}$. In this way we obtain a system of real stochastic equations which can be efficiently numerically simulated:

$$\begin{aligned} dX_\alpha &= \Big\{\Delta Y_\alpha - g\sqrt{2}(Y_\alpha + Y_\beta)X_\gamma - \frac{\kappa_a}{2}X_\alpha\Big\}dt + dW^a_x, \\ dY_\alpha &= \Big\{-\Delta X_\alpha + g\sqrt{2}(X_\alpha + X_\beta)X_\gamma - \frac{\kappa_a}{2}Y_\alpha\Big\}dt + dW^a_y, \\ dX_\beta &= \Big\{-g\sqrt{2}(Y_\alpha + Y_\beta)X_\gamma - \frac{\kappa_b}{2}X_\beta\Big\}dt + dW^b_x, \\ dY_\beta &= \Big\{+g\sqrt{2}(X_\alpha + X_\beta)X_\gamma - \frac{\kappa_b}{2}Y_\beta\Big\}dt + dW^b_y, \\ dX_\gamma &= \Big\{\omega_c Y_\gamma - \frac{\kappa_c}{2}X_\gamma\Big\}dt + dW^c_x, \\ dY_\gamma &= \Big\{-\omega_c X_\gamma + \frac{g}{\sqrt{2}}\left[(X_\alpha+X_\beta)^2 + (Y_\alpha+Y_\beta)^2\right] - \frac{\kappa_c}{2}Y_\gamma\Big\}dt + dW^c_y, \end{aligned} \tag{S5}$$

We then fix $dt = 10^{-3}/\omega_c$ ($1/\omega_c$ being the smallest timescale in the system). Starting with initial conditions $X_\alpha(0) = Y_\alpha(0) = X_\beta(0) = Y_\beta(0) = X_\gamma(0) = Y_\gamma(0) = 0$, we add 10^7 subsequent increments so that the total evolution time becomes $T = 10^7 dt = 10^4/\omega_c \gg 1/\kappa_a, 1/\kappa_b, 1/\kappa_c$ and the final points are distributed consistently with the stationary state of the system. Collecting 10^4 different trajectories, we can finally reconstruct the steady-state distribution in phase-space (which is shown in Fig. 2 in the main text) and extract all the desired statistics.

In Fig. S1 we plot five of these classical trajectories, simulated for $N_a = 0$, $N_b = 0.5$ and $N_c = 0$ (this corresponds to the rightmost column of Fig. 2 in the main text, where all other parameters are also specified).

step time (dt)	$10^{-3}/\omega_c$
number of steps per trajectory	10^7
number of trajectories	10^4

TABLE I: Parameters used in the simulation of the classical stochastic equations. Other system parameters are specified in the main text.

To highlight the asymptotic regime, only the last 20000 points are plotted. Since the frequency of the oscillator b is brought to zero in the rotating frame, the Brownian nature of the motion becomes evident. On the contrary, the oscillator c clearly shows limit cycles of fixed amplitude and random phase.

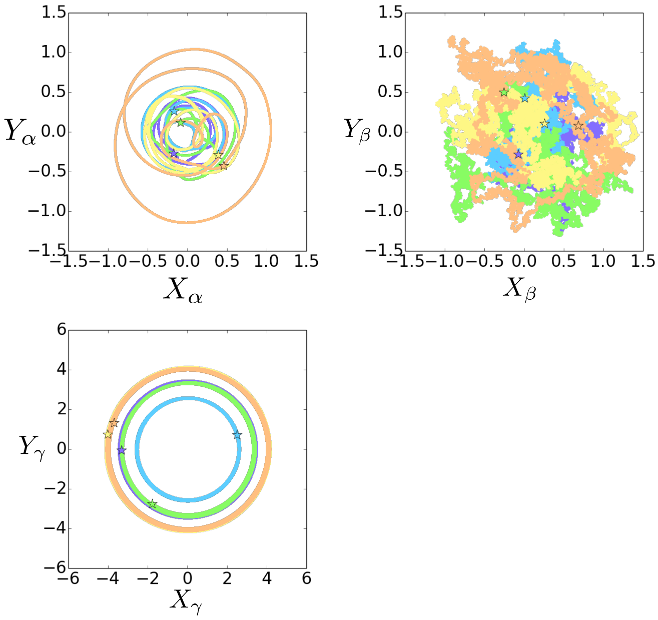

FIG. S1: Five different simulations of the classical stochastic equations (S5), with $N_a = 0$, $N_b = 0.5$ and $N_c = 0$ (corresponding to the rightmost column of Fig. 2 in the main text). For each trajectory, only the last 20000 points (i.e. the asymptotic regime) are shown. Final points are marked by a star. Other parameters are specified in the main text.

[S1] C. W. Gardiner, *Handbook of stochastic methods for physics, chemistry, and the natural sciences*, (Springer, Berlin, 1994)