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Abstract

We study the evolution of geometric quantum discord (GMQD) of a two qubits system coupled with two independent bosonic reservoirs. We consider sub-ohmic, ohmic and super-ohmic. A special attention is devoled to Dicke states and their superpositions.

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1 Introduction

Entanglement has been attracting much attention from physicists in both theory and experiment [1][2][3][4][5], as it plays an important role in quantum information processing such as quantum communications [2] and quantum cryptography [6]. However, there are some exceptions. For instance, quantum computation [7] based on one pure qubit does not employ entanglement, but it needs other non-classical resources. This suggests that entanglement is not the unique resource but just one kind of quantum correlation. In fact, there is another kind of quantum correlation named quantum discord.

Quantum discord, as a measure of bipartite non-classical correlation, is a promising candidate and has generated a lot of interest. For a quantum state ρ in a composite Hilbert space $H = H_A \otimes H_B$, the total amount of correlation can be quantified by quantum mutual information [8]:

$$I(\rho_{AB}) = H(\rho_A) + H(\rho_B) - H(\rho_{AB}) \tag{1}$$

with $H(\rho) = -Tr[\rho \log_2 \rho]$ the von Neumann entropy and $\rho_{A(B)} = Tr_{B(A)}\rho$ the reduced density matrix by tracing system $B(A)$.

Quantum discord is a measure of nonclassical correlations that may include entanglement but is an independent measure. We will document with simple examples that the amounts of classical correlation, quantum discord and entanglement bear no simple relationship to each other. Taking system A as the apparatus, the quantum discord is defined as follows:

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) \tag{2}$$

where $C(\rho_{AB})$ denotes the classical correlations of the state.

The optimization procedure involved in the calculation of quantum discord prevents one to write an analytical expression for quantum discord even for simple two-qubit systems. Quantum discord is analytically computed only for a few families of states including the Bell-diagonal states [9][10], two-qubit X states [11][12], two-qubit rank-2 states [13], a class of rank-2 states of $4 \otimes 2$ systems [14], and Gaussian states of the continuous variable systems [15]. Moreover, based on the optimization of the conditional entropy, an algorithm to calculate the quantum discord of the two-qubit states is presented in . It is also important to have some computable bounds on the quantum discord and some authors have obtained such bounds [16],[17].

This paper is organized as follows. In section 2, we give general expression of GMQD for two qubits in Bosonic Reservoirs and discuss the GMQD for a kind of X-state. In section 3, the Geometric quantum discord. We study in section 4, the GMQD for Reservoirs with Ohmic-Like spectral densities specially the GMQD for Dicke states and their superpositions (generalized W states and GHZ states, the superpositions of two Dicke states). In section 4, the conclusion.

2 General Expression of GMQD for two qubits in bosonic Reservoirs

The model under consideration is a quantum system of two qubits coupled to two independant bosonic reservoirs, the Hamiltonian of wich may be expressed as

$$H = \sum_{j=1}^{2} [\frac{\nu_j}{2}\sigma_{j,3} + \sum_k \omega_{j,k} b_{j,k}^\dagger b_{j,k} + \sum_k \sigma_{j,3}(g_{j,k} b_{j,k}^\dagger + g_{j,k}^* b_{j,k}], \tag{3}$$

where ν_j is the energy difference between the excited state $|1\rangle_j$ and the ground state $|0\rangle_j$, and $\sigma_{j,3}$ is the Pauli matrix of qubit j with $\sigma_{j,3}|1\rangle_j = |1\rangle_j$ and $\sigma_{j,3}|0\rangle_j = -|0\rangle_j$. $b_{j,k}^\dagger(b_{j,k})$ and $\omega_{j,k}$ denote the bosonic creation (annihilation) operator and the frequency of the kth mode of the reservoir of the qubit j,respectively.$g_{j,k}$ denotes the coupling strength between the qubit j and the kth mode. The Hamiltonian describes the spin-boson model without tunneling. A possible experiment setup is a double-dot charge qubit placed in a freestanding semiconductor slab.

Suppose that the two qubits are initially writes in the X states,

$$\rho_s(0) = \frac{1}{4}(I \otimes I + x_3\sigma_3 \otimes I + y_3 I \otimes \sigma_3 + \sum_{i,j=1}^{3} R_{ij}\sigma_i \otimes \sigma_j), \tag{4}$$

the reservoirs are in thermal equilibrium states at temperature T,

$$\rho_{E_j} = exp(-\beta \sum_k \omega_{j,k} b_{j,k}^\dagger b_{j,k})/Z_{E_j}, \tag{5}$$

and the whole system is in the product state,

$$\rho(0) = \rho_s(0) \otimes \rho_{E_1} \otimes \rho_{E_2} \tag{6}$$

where c_i is a real number with $0 \leq |c_i| \leq 1$ for each i,Z_{E_j} is the partition function of the reservoir j with $Z_{E_j} = Tr(exp(-\beta \sum_k \omega_{j,k} b_{j,k}^\dagger b_{j,k})) = \prod_k (1 - e^{-\beta\omega_{j,k}})^{-1}$, and $\beta = 1/(k_B T)$. The whole systems state at time t is governed by

$$\rho(t) = exp(-iHt)\rho(0)exp(iHt), \tag{7}$$

and the state of the two qubits at time t can be obtained by the partial trace

$$\rho_s(t) = Tr_E[exp(-iHt)\rho(0)exp(iHt)]. \tag{8}$$

In order to calculate GMQD, we need to write $\rho_s(t)$ in the form of Eq [?]. From Eq [?], the elements of $\rho_s(t)$ can be expressed as

$$\langle mm'| \rho_s(t) |nn'\rangle = Tr[\xi_1^{mn}(t)\xi_1^{m'n'}(t)\rho_(0)]. \tag{9}$$

where $\xi_1^{mn}(t) = e^{iHt}\xi_1^{mn}e^{-iHt}$ is the Heisenberg operator of qubit j, and $\xi_1^{mn} = |n\rangle_j \langle m|$. Noting that $[\xi_1^{mn}, H] = 0$, we have $\xi_1^{mn}(t) = \xi_1^{mn}$. The operators $\xi_1^{mn}(t)$ for $m \neq n$ can be obtained by solving the Heisenberg equations of motion,

$$i\frac{d}{dt}b_{j,k} = \omega_{j,k}b_{j,k}(t) + g_{j,k}\sigma_{j,3}, \tag{10}$$

$$i\frac{d}{dt}\xi_j^{01}(t) = -\nu_j\xi_j^{01}(t) - 2\sum_k[g_{j,k}b_{j,k}^{\dagger}(t) + g_{j,k}^{*}b_{j,k}(t)]\xi_j^{01}(t) \tag{11}$$

The solutions to the above differential equations are

$$b_{j,k}(t) = e^{-i\omega_{j,k}t}[b_{j,k} + \frac{1}{2}\alpha_{j,k}(t)\sigma_{j,3}] \tag{12}$$

$$\xi_j^{01}(t) = \xi_j^{01}exp\left\{i\nu_j t - \sum_k[\alpha_{j,k}(t)b_{j,k}^{\dagger} - \alpha_{j,k}(t)b_{j,k}]\right\}, \tag{13}$$

with

$$\alpha_{j,k} = 2g_{j,k}(1 - e^{i\omega_{j,k}(t)})/\omega_{j,k}. \tag{14}$$

with the help of Eqs.(8) and (12),we finally obtain

$$\rho_s(t) = \frac{1}{4}(I\otimes I + T_{30}\sigma_3\otimes I + T_{03}I\otimes\sigma_3 + T_{11}\sigma_1\otimes\sigma_1 + T_{12}\sigma_1\otimes\sigma_2 + T_{21}\sigma_2\otimes\sigma_1 + T_{22}\sigma_2\otimes\sigma_2 + T_{33}\sigma_3\otimes\sigma_3), \tag{15}$$

Where

$$T_{11} = [R_{11}cos(\nu_1 t)cos(\nu_2 t) + R_{22}sin(\nu_1 t)sin(\nu_2 t) - R_{12}cos(\nu_1 t)sin(\nu_2 t) - R_{21}sin(\nu_1 t)cos(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{12} = [R_{11}cos(\nu_1 t)sin(\nu_2 t) - R_{22}sin(\nu_1 t)cos(\nu_2 t) + R_{12}cos(\nu_1 t)cos(\nu_2 t) - R_{21}sin(\nu_1 t)sin(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{21} = [R_{11}sin(\nu_1 t)cos(\nu_2 t) - R_{22}cos(\nu_1 t)sin(\nu_2 t) - R_{12}sin(\nu_1 t)sin(\nu_2 t) + R_{21}cos(\nu_1 t)cos(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{22} = [R_{11}sin(\nu_1 t)sin(\nu_2 t) + R_{22}cos(\nu_1 t)cos(\nu_2 t) + R_{12}sin(\nu_1 t)cos(\nu_2 t) + R_{21}cos(\nu_1 t)sin(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{30} = x_3$$

$$T_{03} = y_3$$

$$T_{33} = R_{33} \tag{16}$$

and

$$\gamma(t) = \sum_{j=1}^{2}\sum_k 4\left|g_{j,k}\right|^2\omega_{j,k}^{-2}coth(\frac{\beta\omega_{j,k}}{2})[1 - cos(\omega_{j,k}t)]. \tag{17}$$

$$\rho_s(t) = \begin{pmatrix} \rho_{00}(t) & 0 & 0 & \rho_{03}(t) \\ 0 & \rho_{11}(t) & \rho_{12}(t) & 0 \\ 0 & \rho_{21}(t) & \rho_{22}(t) & 0 \\ \rho_{30}(t) & 0 & 0 & \rho_{33}(t) \end{pmatrix} \tag{18}$$

where

$$\rho_{00}(t) = 1 + T_{30} + T_{03} + T_{33} = \rho_{00}$$

$$\rho_{03}(t) = T_{11} - T_{22} - iT_{12} - iT_{21} = e^{-\gamma(t)}\rho_{03}[cos(\nu_1 t + \nu_2 t)) - isin(\nu_1 t + \nu_2 t)]$$

$$\rho_{11}(t) = 1 + T_{30} - T_{03} - T_{33} = \rho_{11}$$

$$\rho_{12}(t) = T_{11} + T_{22} + iT_{12} - iT_{21} = e^{-\gamma(t)}\rho_{12}[cos(\nu_1 t - \nu_2 t) - isin(\nu_1 t - \nu_2 t)]$$

$$\rho_{21}(t) = T_{11} + T_{22} - iT_{12} + iT_{21} = e^{-\gamma(t)}\rho_{21}[cos(\nu_1 t - \nu_2 t) + isin(\nu_1 t - \nu_2 t)]$$

$$\rho_{22}(t) = 1 - T_{30} + T_{03} - T_{33} = \rho_{22}$$

$$\rho_{30}(t) = T_{11} - T_{22} + iT_{12} + iT_{21} = e^{-\gamma(t)}\rho_{30}[cos(\nu_1 t + \nu_2 t) + isin(\nu_1 t + \nu_2 t)]$$

$$\rho_{33}(t) = 1 - T_{30} - T_{03} + T_{33} = \rho_{33} \tag{19}$$

These matrix elements can be written also as follows

$$\rho_{00}(t) = \rho_{00}$$

$$\rho_{03}(t) = e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}\rho_{03}$$

$$\rho_{11}(t) = \rho_{11}$$

$$\rho_{12}(t) = e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}\rho_{12}$$

$$\rho_{21}(t) = e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}\rho_{21}$$

$$\rho_{22}(t) = \rho_{22}$$

$$\rho_{30}(t) = e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}\rho_{30}$$

$$\rho_{33}(t) = \rho_{33} \tag{20}$$

3 Geometric quantum discord

Dakic et al defined the geometric measure of quantum discord [**?**]as the distance between a state ρ of a bipartite system AB and the closest classical-quantum state presenting zero discord:

$$D_g(\rho) := \min_{\chi} ||\rho - \chi||^2 \tag{21}$$

where the minimum is over the set of zero-discord states χ and the distance is the square norm in the Hilbert-Schmidt space. It is given by

$$||\rho - \chi||^2 := \mathrm{Tr}(\rho - \chi)^2.$$

When the measurement is taken on the subsystem A, the zero-discord state χ can be represented as [**?**]

$$\chi = \sum_{i=1,2} p_i|\psi_i\rangle\langle\psi_i| \otimes \rho_i$$

where p_i is a probability distribution, ρ_i is the marginal density matrix of B and $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an arbitrary orthonormal vector set.

For an arbitrary two-qubit state, the density matrix ρ can be expressed as

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} x_i \sigma_i \otimes I + \sum_{i=1}^{3} I \otimes y_i \sigma_i + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j), \tag{22}$$

where $x_i = Tr(\rho \sigma_i \otimes I)$, $y_i = Tr(\rho I \otimes \sigma_i)$ are the components of local Bloch vectors, and $T_{ij} = Tr(\rho \sigma_i \otimes \sigma_j)$ are the elements of the correlation matrix. The operators $\sigma_i (i = 1, 2, 3)$ stand for the three Pauli matrices and σ_0 is the identity matrix. For the state described by Eq(4), GMQD can be explicity expressed by the formula,[10]

$$D_g(\rho) = \frac{1}{4}\left(||x||^2 + ||R||^2 - \lambda_{\max}\right) \tag{23}$$

where $x = (0, 0, x_3)^T$, R is the matrix with elements R_{ij}, and $\lambda_{\max}$ is the largest eigenvalue of matrix defined by

$$K := xx^T + RR^T. \tag{24}$$

Denoting the eigenvalues of the 3×3 matrix K by λ_1, λ_2 and λ_3 and considering $||x||^2 + ||R||^2 = \mathrm{Tr}K$, we get an alternative compact form of the geometric measure of quantum discord

$$D_g(\rho) = \frac{1}{4} \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \tag{25}$$

This form will be more convenient for our purpose. For the density matrix under consideration, the eigenvalues of the matrix k are given by

$$\lambda_1 \equiv \lambda_1(t) = 4(|\rho_{12}(t)| + |\rho_{03}(t)|)^2$$

$$\lambda_2 \equiv \lambda_2(t) = 4(|\rho_{12}(t)| - |\rho_{03}(t)|)^2$$

$$\lambda_3 \equiv \lambda_3(t) = 2\left[(\rho_{00}(t) - \rho_{22}(t))^2 + (\rho_{11}(t) - \rho_{33}(t))^2\right]$$

finally, we find:

$$\lambda_1(t) = 4(\left|e^{-\gamma(t)} e^{-i(v_1 - v_2)t} \rho_{12}\right| + \left|e^{-\gamma(t)} e^{-i(v_1 + v_2)t} \rho_{03}\right|)^2 = e^{-2\gamma(t)} \lambda_1(0)$$
$$\lambda_2(t) = 4(\left|e^{-\gamma(t)} e^{-i(v_1 - v_2)t} \rho_{12}\right| + \left|e^{-\gamma(t)} e^{-i(v_1 + v_2)t} \rho_{03}\right|)^2 = e^{-2\gamma(t)} \lambda_2(0)$$
$$\lambda_3(t) = 2[(\rho_{00} - \rho_{22})^2 + (\rho_{11} - \rho_{33})^2] = \lambda_3(0)$$

Clearly, the eigenvalue λ_1 is larger than λ_2.

4 GMQD for Reservoirs with Ohmic-Like spectral Densities

Equation (25) gives the GMQD of two qubits in the bosonic reservoirs. $D[\rho_s(t)]$ is a time-dependent function through the parameter $\gamma(t)$ defined by Eq (17). In the continuum limit, $\sum_k 4g_{j,k}^2$ can be replaced by $\int d\omega J_j(\omega)\delta(\omega_{j,k}-\omega)$. We further let $J_1(\omega)=J_2(\omega)=J(\omega)$, which means that the two resevoirs have the same spectral density. In this case, there is

$$\gamma(t) = 2\int_0^\infty d\omega J(\omega)\omega^{-2}\coth(\beta\omega/2)[1-\cos(\omega t)]. \tag{26}$$

For Ohmic-like spectral densities, $J(\omega)$ reads

$$J(\omega) = \lambda\Omega^{1-s}\omega^s e^{-\omega/\Omega} \tag{27}$$

where λ is a dimensionless coupling constant and Ω is the cutoff frequency. $0<s<1$, $s=1$, and $s>1$ are respectively corresponding to sub-Ohmic, Ohmic, and super-Ohmic reservoirs. In the following paragraphs, we discuss the evolution of $D[\rho_s(t)]$ in three kinds of Ohmic-like reservoirs, respectively.

4.1 Sub-Ohmic reservoirs

For the case of sub-ohmic reservoirs, $0<s<1$, the integral (26) can be calculated

$$\gamma(t) = \frac{2\lambda\Gamma(s)}{s-1}\left\{1-(1+\Omega^2t^2)^{(1-s)/2}cos[(s-1)arctan(\Omega t)]\right\}$$

$$+\frac{4\lambda\Gamma(s)}{s-1}\sum_{m=1}^{\infty}(1+m\Omega\beta)^{1-s}\times\left\{1-[1+(\frac{\Omega t}{1+m\Omega\beta})^2]^{(1-s)/2}\times cos[(s-1)arctan(\frac{\Omega t}{1+m\Omega\beta})]\right\} \tag{28}$$

where $\Gamma(s)$ is the function. To investigate the evolution of $D[\rho_s(t)]$, we calculate the derivative of $\gamma(t)$ for time t,

$$\frac{\partial\gamma(t)}{\partial t} = 2\lambda\Gamma(s)\Omega\left\{(1+\Omega^2t^2)^{-s/2}\sin[sarctan(\Omega t)]+2\sum_{m=1}^{\infty}[(1+m\Omega\beta)^2+\Omega^2t^2]^{-s/2}\times\sin[sarctan(\frac{\Omega t}{1+m\Omega\beta})]\right\}. \tag{29}$$

Since $0<\arctan[\Omega t/(1+m\Omega\beta)]<\arctan(\Omega t)<\pi/2$, there $\partial\gamma(t)/\partial t>0$ for all time t. Hence, $\gamma(t)$ is a monotonic increasing function of t, and it tends to infinity as t goes to infinity.

4.2 Ohmic reservoirs

For the case of Ohmic reservoirs, s=1, the integral (26)can be worked out as

$$\gamma(t) = \lambda[ln(1+\Omega^2t^2)+4ln\Gamma(1+\frac{1}{\Omega t})-2ln\left|\Gamma(1+\frac{1}{\Omega\beta}+i\frac{t}{\beta})\right|^2] \tag{30}$$

The derivative of $\gamma(t)$ is

$$\frac{\partial\gamma(t)}{\partial t} = 2\lambda\Omega^2t[\frac{1}{1+\Omega^2t^2}+\sum_{m=1}^{\infty}\frac{2}{(1+m\Omega\beta)^2+\Omega^2t^2}]. \tag{31}$$

Since $\partial\gamma(t)/\partial t > 0$, $\gamma(t)$ is monotonic increasing for all time t,and it tends to infinity as T goes to infinity.

4.3 Super-Ohmic reservoirs

For the case of super-Ohmic reservoirs, $s > 1$, the integral (26) can be worked out as

$$\gamma(t) = 2\lambda\Gamma(s-1)\left\{1-(1+\Omega^2t^2)^{(1-s)/2}\right\} \times cos[(s-1)arctan(\Omega t)$$
$$+4\lambda\Gamma(s-1)\sum_{m=1}^{\infty}(1+m\Omega\beta)^{(1-s)} \times \left\{1-[1+(\frac{\Omega t}{1+m\Omega\beta})^2]^{(1-s)/2} \times cos[(s-1)arctan(\frac{\Omega t}{1+m\Omega\beta})]\right\} \tag{32}$$

and the time derivative of it reads

$$\frac{\partial\gamma(t)}{\partial t} = 2\lambda\Gamma(s)\Omega\left\{(1+\Omega^2t^2)^{-s/2}\sin[sarctan(\Omega t)]+2\sum_{m=1}^{\infty}[(1+m\Omega\beta)^2+\Omega^2t^2]^{-s/2} \times \sin[sarctan(\frac{\Omega t}{1+m\Omega\beta})]\right\}. \tag{33}$$

Since $0 < \arctan[\Omega t/(1+m\Omega\beta)] < \arctan(\Omega t) < \pi/2$, there $\partial\gamma(t)/\partial t > 0$ for all time t. Hence, $\gamma(t)$ is a monotonic increasing function of t, and it tends to infinity as t goes to infinity.

4.4 GMQD for dicke and their superpositions

we consider a state of two qubits which have been extracted from the whole ensemble. Specifically, we concentrate on the states with exchange symmetry and parity, whose two-qubit reduced density matrix can be written as

$$\rho^{AB} = \begin{pmatrix} \upsilon_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & \upsilon_- \end{pmatrix} \tag{34}$$

in the basis of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, with υ_+, υ_-. and y real, and u^*. the complex conjugate of u. The elements of the density matrix can be represented by the expectation values of the spin components

$$\upsilon_\pm = \frac{N^2-2N+4\langle J_z^2\rangle \pm 4\langle J_z\rangle(N-1)}{4N(N-1)}, \tag{35}$$

$$y = \frac{N^2-4\langle J_z^2\rangle}{4N(N-1)}, u = \frac{\langle J_+^2\rangle}{N(N-1)} \tag{36}$$

4.4.1 Dicke states

The so-called Dicke states under consideration are defined as

$$|n\rangle_N \equiv |N/2, -N/2+n\rangle, \qquad n = 0, ..., N, \tag{37}$$

As we have discussed in the previous section, to obtain the GMQD, we need to calculate the expectation values of the spin components for the stats $|n\rangle_N$, which are

$$\begin{aligned} \langle J_z \rangle &= n - \frac{N}{2}, \\ \langle J_z^2 \rangle &= (n - \frac{N}{2})^2, \\ \langle J_+^2 \rangle &= \langle J_-^2 \rangle = 0. \end{aligned} \tag{38}$$

Two-qubit reduced density matrix can be written as

$$\rho^{AB} = \begin{pmatrix} \upsilon_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & \upsilon_- \end{pmatrix} \tag{39}$$

Where:

$$\upsilon_+ = \frac{N^2 - 2N + 4\langle J_z^2 \rangle + 4\langle J_z \rangle (N-1)}{4N(N-1)} = \frac{n(n-1)}{N(N-1)}, \tag{40}$$

$$\upsilon_- = \frac{N^2 - 2N + 4\langle J_z^2 \rangle - 4\langle J_z \rangle (N-1)}{4N(N-1)} = 1 + \frac{n^2 - 2nN + n}{N(N-1)}, \tag{41}$$

$$\begin{aligned} y &= \frac{N^2 - 4\langle J_z^2 \rangle}{4N(N-1)} = \frac{-n^2 + nN}{N(N-1)}, \\ u &= \frac{\langle J_+^2 \rangle}{N(N-1)} = 0 \end{aligned} \tag{42}$$

Using the formula (19) we can write the density matrix for each system at t after intercation.

$$\begin{aligned} \rho_{00}(t) &= \rho_{00} = \upsilon_+ = \frac{n(n-1)}{N(N-1)} \\ \rho_{03}(t) &= e^{-\gamma(t)} e^{-i(\nu_1+\nu_2)t} \rho_{03} = e^{-\gamma(t)} e^{-i(\nu_1+\nu_2)t} u^* = 0 \\ \rho_{11}(t) &= \rho_{11} = y = \frac{-n^2 + nN}{N(N-1)} \\ \rho_{12}(t) &= e^{-\gamma(t)} e^{-i(\nu_1-\nu_2)t} \rho_{12} = e^{-\gamma(t)} e^{-i(\nu_1-\nu_2)t} y = e^{-\gamma(t)} e^{-i(\nu_1-\nu_2)t} \frac{-n^2 + nN}{N(N-1)} \\ \rho_{21}(t) &= e^{-\gamma(t)} e^{i(\nu_1-\nu_2)t} \rho_{21} = e^{-\gamma(t)} e^{i(\nu_1-\nu_2)t} y = e^{-\gamma(t)} e^{i(\nu_1-\nu_2)t} \frac{-n^2 + nN}{N(N-1)} \\ \rho_{22}(t) &= \rho_{22} = y = \rho_{22} = \frac{-n^2 + nN}{N(N-1)} \\ \rho_{30}(t) &= e^{-\gamma(t)} e^{i(\nu_1+\nu_2)t} \rho_{30} = e^{-\gamma(t)} e^{i(\nu_1+\nu_2)t} u = 0 \\ \rho_{33}(t) &= \rho_{33} = \upsilon_- = 1 + \frac{n^2 - 2nN + n}{N(N-1)} \end{aligned} \tag{43}$$

This form will be more convenient for our purpose. For the density matrix under consideration, the eigenvalues of the matrix k are given by

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_3 \equiv \lambda_3(t) = \lambda_3(0) = \frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2-N]^2}{N^2(N-1)^2}$$

$\lambda_1(t) = \lambda_2(t)$ and the maximum of the eigenvalues is $\lambda_1(t)$ or $\lambda_3(t)$

Therefore, for these X-states, equation (23) is reduced to a simpler form

$$D_G(\rho) = \frac{1}{4}(min\{\lambda_1(t), \lambda_3(t)\} + \lambda_2(t)) \tag{44}$$

Since they are functions of N and n, we will discuss the properties of the GMQD in one case with N and n fixed.

4.4.2 Generalized GHZ states

Now we discuss the GMQD of the generalized GHZ states which are superpositions of two special Dicke states. The generalized GHZ states which are important in the research of quantum mechanics are states in the following form:

$$|\psi\rangle_{GHZ} = cos\theta\,|0\rangle_N + e^{i\phi}sin\theta\,|N\rangle_N\,. \tag{45}$$

For such states, the spin expectation values are

$$\begin{aligned}\langle J_z\rangle &= -\frac{N}{2}cos2\theta,\\ \langle J_z^2\rangle &= \frac{N^2}{4},\\ \langle J_+^2\rangle &= 0.\end{aligned} \tag{46}$$

Two-qubit reduced density matrix can be written as

$$\rho^{AB}\begin{pmatrix} \upsilon_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & \upsilon_- \end{pmatrix} \tag{47}$$

Where:

$$\upsilon_+ = \frac{N^2 - 2N + 4\langle J_z^2\rangle + 4\langle J_z\rangle(N-1)}{4N(N-1)} = \frac{N - Ncos(2\theta) - 1 + cos(2\theta)}{2(N-1)}, \tag{48}$$

$$\upsilon_- = \frac{N^2 - 2N + 4\langle J_z^2\rangle - 4\langle J_z\rangle(N-1)}{4N(N-1)} = \frac{N + Ncos(2\theta) - 1 - cos(2\theta)}{2(N-1)}, \tag{49}$$

$$y = \frac{N^2 - 4\left\langle J_z^2 \right\rangle}{4N(N-1)} = 0,$$

$$u = \frac{\left\langle J_+^2 \right\rangle}{N(N-1)} = 0. \tag{50}$$

Using the formula (19) we can write the density matrix for each system at t after intercation.

$$\rho_{00}(t) = \rho_{00} = v_+ = \frac{N - Ncos(2\theta) - 1 + cos(2\theta)}{2(N-1)}$$

$$\rho_{03}(t) = e^{-\gamma(t)} e^{-i(\nu_1+\nu_2)t} \rho_{03} = e^{-\gamma(t)} e^{-i(\nu_1+\nu_2)t} u^* = 0$$

$$\rho_{11}(t) = \rho_{11} = y = 0$$

$$\rho_{12}(t) = e^{-\gamma(t)} e^{-i(\nu_1-\nu_2)t} \rho_{12} = e^{-\gamma(t)} e^{-i(\nu_1-\nu_2)t} y = 0$$

$$\rho_{21}(t) = e^{-\gamma(t)} e^{i(\nu_1-\nu_2)t} \rho_{21} = e^{-\gamma(t)} e^{i(\nu_1-\nu_2)t} y = 0$$

$$\rho_{22}(t) = \rho_{22} = y = 0$$

$$\rho_{30}(t) = e^{-\gamma(t)} e^{i(\nu_1+\nu_2)t} \rho_{30} = e^{-\gamma(t)} e^{i(\nu_1+\nu_2)t} u = 0$$

$$\rho_{33}(t) = \rho_{33} = v_- = \frac{N + Ncos(2\theta) - 1 - cos(2\theta)}{2(N-1)} \tag{51}$$

This form will be more convenient for our purpose. For the density matrix under consideration, the eigenvalues of the matrix k are given by

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)} \lambda_1(0) = 0$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)} \lambda_2(0) = 0$$

$$\lambda_3 \equiv \lambda_3(t) = 2\left[(\rho_{00}(t) - \rho_{22}(t))^2 + (\rho_{11}(t) - \rho_{33}(t))^2\right] = \lambda_3(0) = 1 + cos^2 2\theta$$

which lead to

$$D_g(\rho) = 0 \tag{52}$$

4.4.3 Superpositions of Dicke states

In this subsection, weinvestigate a more general superposition of Dicke states, which reads

$$|\psi\rangle_{SD} = cos\theta \, |n\rangle_N + e^{i\phi} sin\theta \, |n+2\rangle_N \,, \qquad n = 0, ..., N-2 \tag{53}$$

with the angle $\theta\epsilon[0,\pi)$ and relative phase $\varphi\epsilon[0,2\pi)$. Then the expressions of the relevant spin expectation values are

$$\begin{aligned}
\langle J_z \rangle &= (n-\frac{N}{2})cos^2\theta(n+2-\frac{N}{2})sin^2\theta, \\
\left\langle J_z^2 \right\rangle &= (n-\frac{N}{2})^2cos^2\theta(n+2-\frac{N}{2})^2sin^2\theta, \\
\left\langle J_+^2 \right\rangle &= \frac{1}{2}e^{i\phi}sin2\theta\sqrt{\mu_n}.
\end{aligned} \tag{54}$$

Where

$$\mu_N = (n+1)(n+2)(N-n)(N-n-1). \tag{55}$$

Two-qubit reduced density matrix can be written as

$$\rho^{AB} \begin{pmatrix} \upsilon_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & \upsilon_- \end{pmatrix} \tag{56}$$

Where:

$$\upsilon_+ = \frac{N^2-2N+4\left\langle J_z^2 \right\rangle+4\left\langle J_z \right\rangle(N-1)}{4N(N-1)} = \frac{n^2+4nsin^2\theta-n+2sin^2\theta}{N(N-1)} \tag{57}$$

$$\upsilon_- = \frac{N^2-2N+4\left\langle J_z^2 \right\rangle-4\left\langle J_z \right\rangle(N-1)}{4N(N-1)} = \frac{N^2-N+n^2-2nN+n+4nsin^2\theta+4sin^2\theta-4Nsin^2\theta}{N(N-1)} \tag{58}$$

$$\begin{aligned}
y = \frac{N^2-4\left\langle J_z^2 \right\rangle}{4N(N-1)} &= \frac{-n^2+nN-4nsin^2\theta-4sin^2\theta+2Nsin^2\theta}{N(N-1)}, \\
u = \frac{\left\langle J_+^2 \right\rangle}{N(N-1)} &= \frac{1}{2}\frac{e^{i\phi}sin2\theta\sqrt{\mu_n}}{N(N-1)}.
\end{aligned} \tag{59}$$

Using the formula (19) we can write the density matrix for each system at t after intercation

$$\rho_{00}(t) = \rho_{00} = \upsilon_+ = \frac{n^2-n+4nsin^2\theta+2sin^2\theta}{N(N-1)}$$

$$\rho_{03}(t) = e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}\rho_{03} = e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}u^* = e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}e^{-i\phi}\frac{sin2\theta}{\sqrt{2N(N-1)}}$$

$$\rho_{11}(t) = \rho_{11} = y = \frac{-n^2+nN-4nsin^2\theta-4sin^2\theta+2Nsin^2\theta}{N(N-1)}$$

$$\rho_{12}(t) = e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}\rho_{12} = e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}y = e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}\frac{-n^2+nN-4nsin^2\theta-4sin^2\theta+2Nsin^2\theta}{N(N-1)}$$

$$\rho_{21}(t) = e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}\rho_{21} = e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}y = e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}\frac{-n^2+nN-4nsin^2\theta-4sin^2\theta+2Nsin^2\theta}{N(N-1)}$$

$$\rho_{22}(t)=\rho_{22}=y=\frac{-n^2+nN-4nsin^2\theta-4sin^2\theta+2Nsin^2\theta}{N(N-1)}$$

$$\rho_{30}(t)=e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}\rho_{30}=e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}u=e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}e^{i\phi}\frac{sin2\theta}{\sqrt{2N(N-1)}}$$

$$\rho_{33}(t)=\rho_{33}=\upsilon_-=\frac{N^2-N+n^2-2nN+n+4nsin^2\theta+6sin^2\theta-4Nsin^2\theta}{N(N-1)} \quad (60)$$

The GMQD for these states can be obtained by repeating the previous processing step. AsDG is not a function of , we discuss the GMQD versus in the following. Specifically, when n = 0

$$\rho_{00}(t)=\rho_{00}=\upsilon_+=\frac{2sin^2\theta}{N(N-1)}$$

$$\rho_{03}(t)=e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}\rho_{03}=e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}u^*=e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}\frac{sin2\theta}{\sqrt{2N(N-1)}}$$

$$\rho_{11}(t)=\rho_{11}=y=\frac{2(N-2)}{N(N-1)}sin^2\theta$$

$$\rho_{12}(t)=e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}\rho_{12}=e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}y=e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}\frac{2(N-2)}{N(N-1)}sin^2\theta$$

$$\rho_{21}(t)=e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}\rho_{21}=e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}y=e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}\frac{2(N-2)}{N(N-1)}sin^2\theta$$

$$\rho_{22}(t)=\rho_{22}=y=\frac{2(N-2)}{N(N-1)}sin^2\theta$$

$$\rho_{30}(t)=e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}\rho_{30}=e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}u=e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}e^{i\phi}\frac{sin2\theta}{\sqrt{2N(N-1)}}$$

$$\rho_{33}(t)=\rho_{33}=\upsilon_-=\frac{N^2-N+6sin^2\theta-4Nsin^2\theta}{N(N-1)} \quad (61)$$

This form will be more convenient for our purpose. For the density matrix under consideration

$$\lambda_1\equiv\lambda_1(t)=e^{-2\gamma(t)}\lambda_1(0)=4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta+\frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_2\equiv\lambda_2(t)=e^{-2\gamma(t)}\lambda_2(0)=4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta-\frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_3\equiv\lambda_3(t)=2\big[(\rho_{00}(t)-\rho_{22}(t))^2+(\rho_{11}(t)-\rho_{33}(t))^2\big]=(1-\frac{4}{N}sin^2\theta)^2+[1-\frac{8(N-2)}{N(N-1)}sin^2\theta]^2$$

`Dicke states`

For the density matrix under consideration, the eigenvalues of the matrix k are given by

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_3 \equiv \lambda_3(t) = \lambda_3(0) = \frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2-N]^2}{N^2(N-1)^2}$$

$\lambda_1(t) = \lambda_2(t)$ and the maximum of the eigenvalues is $\lambda_1(t)$ or $\lambda_3(t)$

Therefore, for these X-states, equation (23) is reduced to a simpler form

$$D_G(\rho) = \frac{1}{4}(min\{\lambda_1(t), \lambda_3(t)\} + \lambda_2(t)) \tag{62}$$

Specifically, When $N = 15 \quad n = 0, ..., 15$

- For $n = 0, ..., 4$ and $n = 11, ..., 15$

  The geometric quantum discord ρ_{12} is

$$D_\rho = \frac{1}{2}(\lambda_1) = \frac{1}{2}e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2} \tag{63}$$

  when $\lambda_3 > \lambda_1$ (*)

- For $n = 5$ and $n = 10$ the condition (*) is satisfied for $t_0 < t$. In this situation, the geometric quantum discord is

$$D_G = \frac{1}{2}e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2} \tag{64}$$

  when the transmission parameter t satisfies $t_0 > t$

  we have $\lambda_3 < \lambda_1$ and the geometric quantum discord is given by

$$D_G = \frac{1}{4}(\lambda_3 + \lambda_2) = \frac{1}{4}(\frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2-N]^2}{N^2(N-1)^2} + e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}) \tag{65}$$

- For $n = 6, 7$ and $n = 8, 9$

  the geometric quantum discord ρ_{12} is

$$D_G = \frac{1}{4}(\lambda_3 + \lambda_2) = \frac{1}{4}(\frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2-N]^2}{N^2(N-1)^2} + e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}) \tag{66}$$

  when$\lambda_3 < \lambda_1$

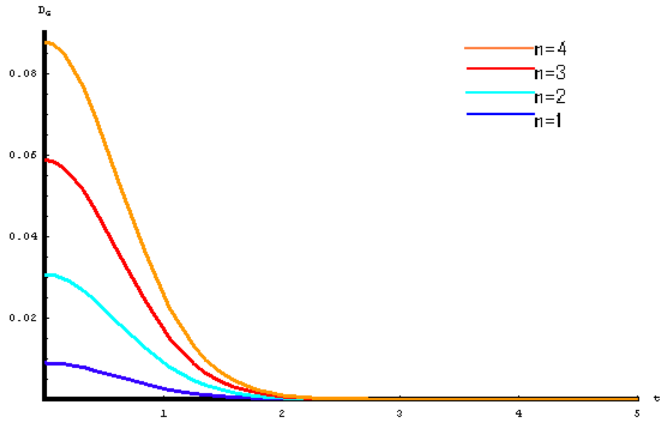
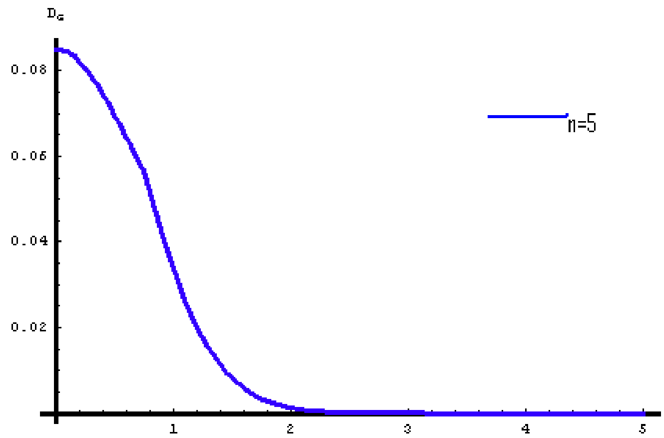
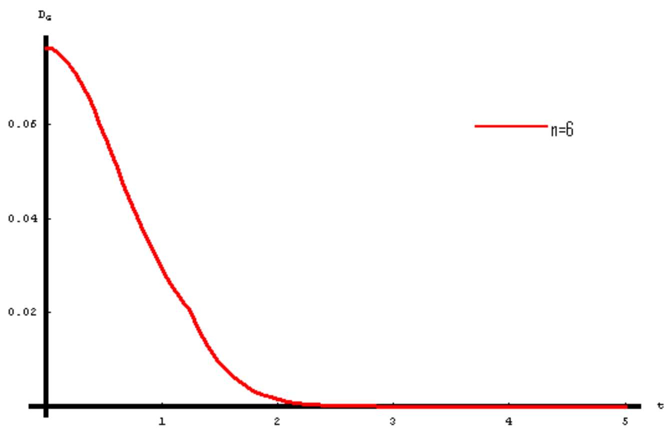


Figure 1: (a) Dynamics of GMQD for the sub-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 1, .., 4$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).





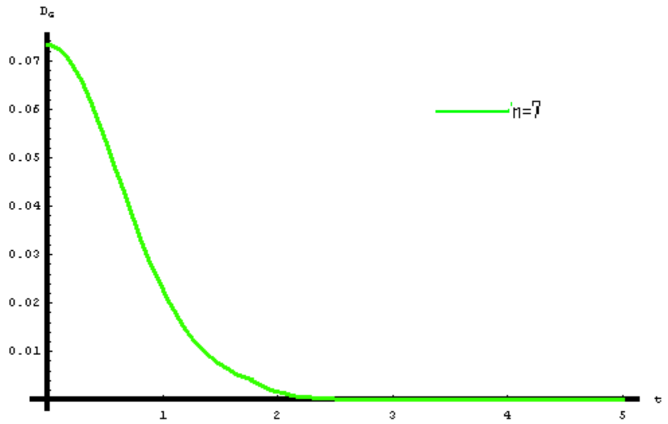


Figure 1: (b) Dynamics of GMQD for the sub-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 5, 6, 7$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).

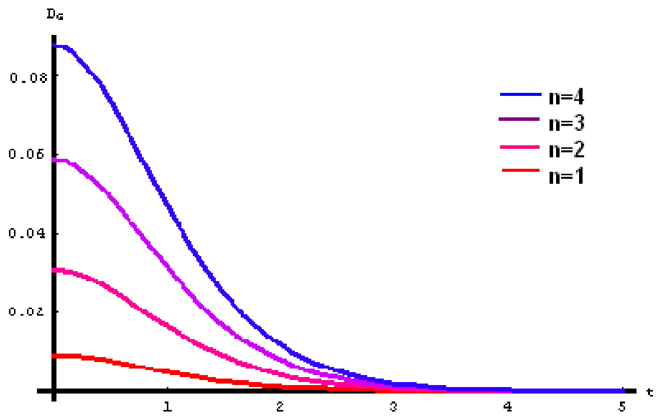
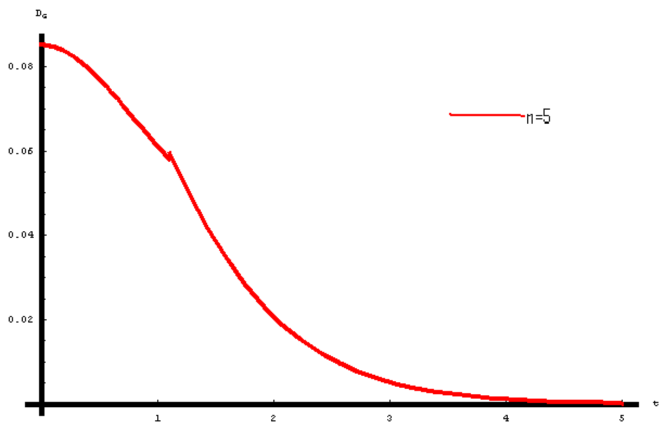


Figure 2: (a) Dynamics of GMQD for the Ohmic reservoirs of dicke states with N fixed ($N = 15$) and ($n = 1, .., 4$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).



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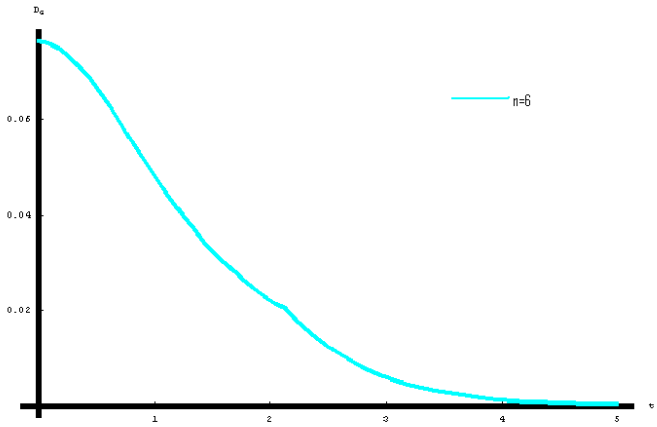

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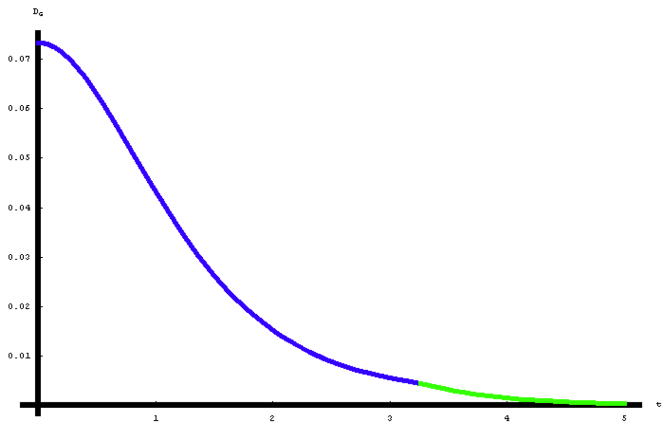

Figure 2: (b) Dynamics of GMQD for the Ohmic reservoirs of dicke states with N fixed ($N = 15$) and ($n = 5, 6, 7$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).

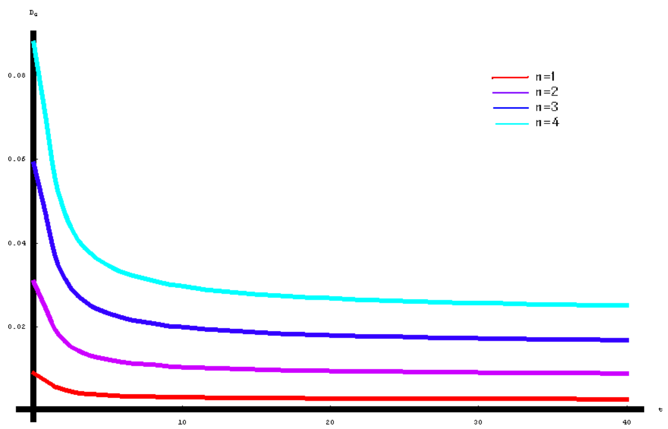

Figure 3: (a) Dynamics of GMQD for the super-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 1, .., 4$) $s = 1.5, \lambda = 0.2$ (numerical calculation with the upper bound of m being 10^5).

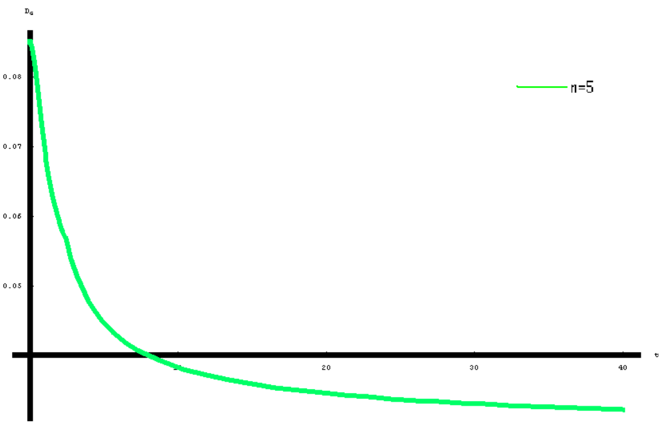

Figure 3: (b) Dynamics of GMQD for the super-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 5$) $s = 1.5, \lambda = 0.2$ (numerical calculation with the upper bound of m being 10^5).

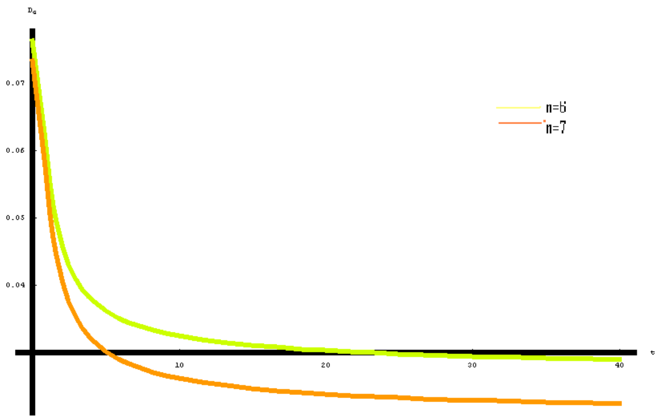

Figure 3: (c)Dynamics of GMQD for the super-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 0, .., 15$) $s = 1.5, \lambda = 0.2$ (numerical calculation with the upper bound of m being 10^5).

It can also be seen that there are two classes of the evolution of GMQD. (a)D_G is a monotonic decreasing function of time t with the limit zero, if $\lambda_3 > \lambda_1$. (b)D_G is a piecewise monotonic decreasing function with one turning point at the time $t = \bar{t}$ defined by $\lambda_3 = \lambda_1$ and then has the limit zero as t tends to infinity, if $\lambda_3 < \lambda_1$. These two classes of the evolution of D_G are plotted in fig 1 and 2.

In the figure 3, there are two possible classes of the evolution of D_G.(a)and (b) D_G is a monotonic decreasing function of time with a constant limit $(1/2)\lambda_1$ or $(1/4)(\lambda_3 + \lambda_2)$, if $\lambda_3 > \lambda_1$, or if $\lambda_3 < \lambda_1$. (b)D_G is a piecewise monotonic decreasing function with one turning point at the time $t = \bar{t}$ defined by $\lambda_3 = \lambda_1$, and then has a constant limit as t tends to infinity, if $\lambda_3 < \lambda_1$.

`Superpositions of Dicke states`

For the density matrix under consideration

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = 4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta + \frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = 4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta - \frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_3 \equiv \lambda_3(t) = 2\left[(\rho_{00}(t) - \rho_{22}(t))^2 + (\rho_{11}(t) - \rho_{33}(t))^2\right] = (1 - \frac{4}{N} sin^2\theta)^2 + [1 - \frac{8(N-2)}{N(N-1)} sin^2\theta]^2 \quad (67)$$

It is noted that D_G is a periodic function of θ with period π. Accordingly, we plot the GMQD versus within one period. To compare the eigenvalues λ_1 and λ_3, we find that there are two cases.

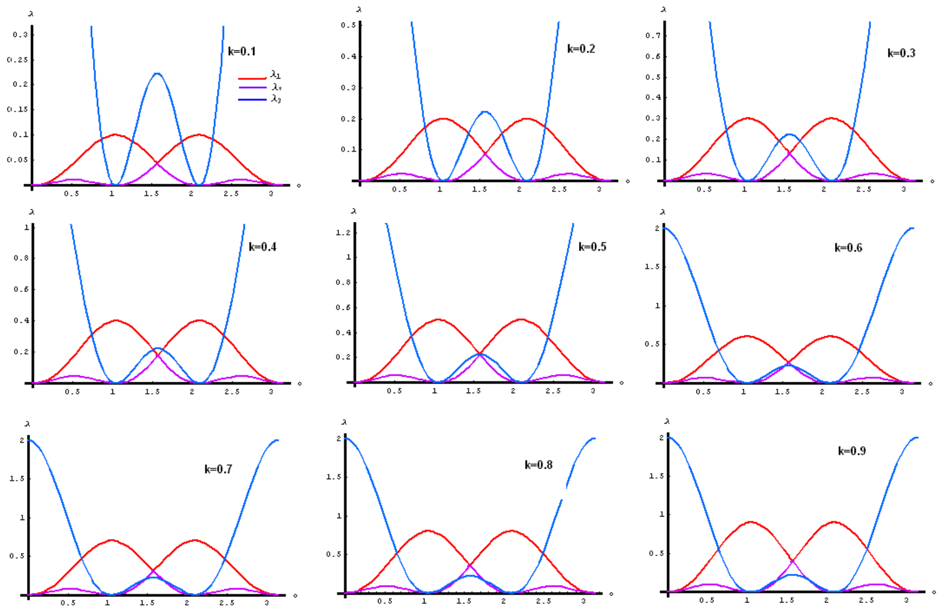

Figure 4: Eigenvalues of K of the superpositions of Dicke states verus θ when $n = 0$ with $N = 3$ and $k = 0.1, ...0.9$.

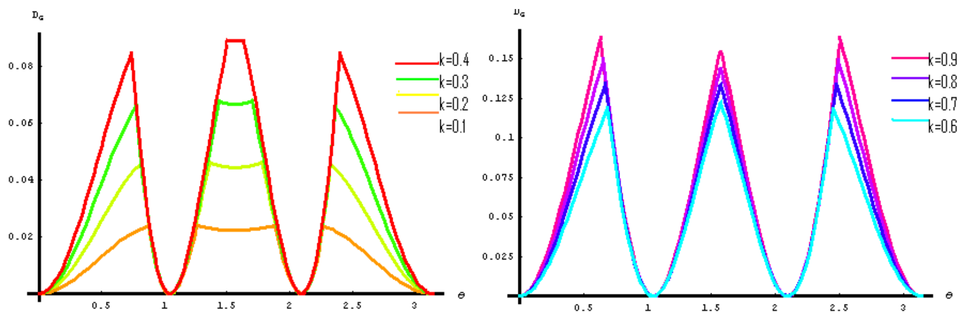

Figure 5: The GMQD of the superpositions of Dicke states verus θ when $n = 0$ with $N = 3$ and $k = 0.1, ...0.9$.

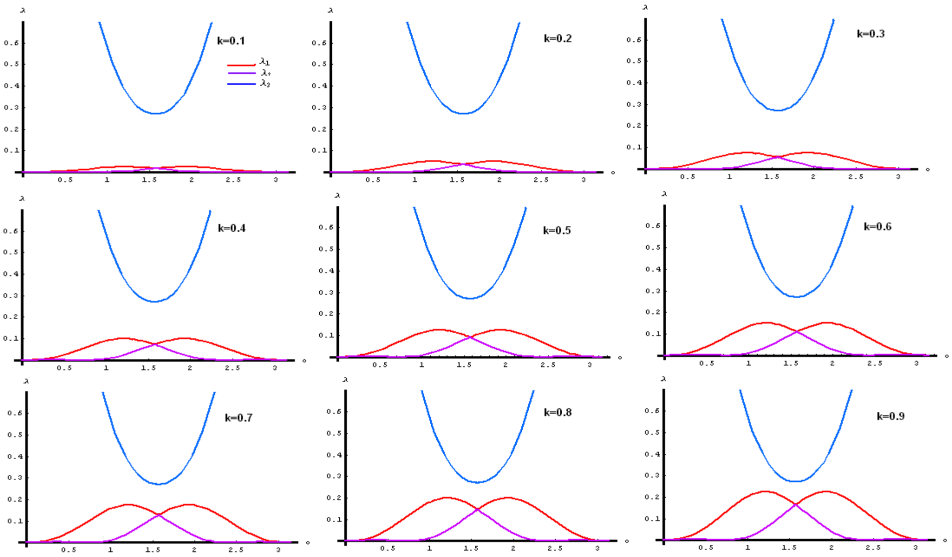

Figure 6: Eigenvalues of K of the superpositions of Dicke states verus θ when $n = 0$ with $N = 8$ and $k = 0.1, ...0.9$.

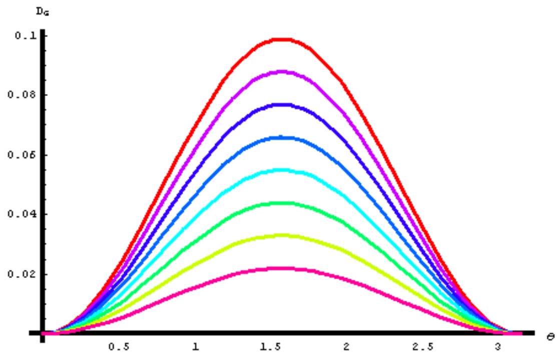

Figure 7: The GMQD of the superpositions of Dicke states verus θ when $n = 0$ with $N = 8$ and $k = 0.1, ...0.9$.

When $N < 8$, as can be seen in figure 4 (N=3), the maximum eigenvalue is λ_1 in the middle region and λ_3 at the edge near 0 and π for $k = 0.6, 0.7, 0.8, 0.9$ and for $k = 0.1, 0.2, 0.3, 0.4$ the maximum eigenvalue is λ_3 in the middle region and at the edge near 0 and π and λ_1 between the middle region and at the edge near 0.

In figure 5, there are three local maxima of D_G. The first and third maxima display just at the point when $\lambda_2 = \lambda_3$. While the second one accurs at $\theta = \frac{\pi}{2}$ which is due to the symmetry of equation (67). We also observe that there exist two special values of θ, $\theta = \pi/3$ and $\theta = 2\pi/3$, where the GMQD is zero. These zero values of the GMQD are induced by $\lambda_2 = \lambda_3 = 0$ at these two points (while λ_1 is the maximum).

However,When $N \geq 8$ as is shown in fig 6, $\lambda_3 > \lambda_1$ no matter what θ is. Consequently, D_G has only one maximum at $\theta = \pi/2$, as is shown in figure 7, and it can analytically be written as

$$D_G = e^{-2\gamma(t)}[\frac{8(N-2)^2 sin^4\theta}{N^2(N-1)^2} + \frac{sin^2 2\theta}{N(N-1)}] \quad (68)$$

5 Conclusion

In this paper, in order to investigate the pairwise quantum correlations, we have studied the dynamics of GMQD The model under consideration consists of two qubits coupled with two independent bosonic reservoirs described by Ohmic-like spectral densities. The Hamiltonian of the model is described as Eq (3), and the two-qubit system is initially in the X states decoupled from the environments, described by Eq (4). We examine the evolution of GMQD for the three types of reservoirs, sub-ohmic, ohmic, and super-ohmic particularly the Dicke states and their superpositions. D_G is a monotonic decreasing function of time t or a poiecewise monotonic decreasing function with one turning point before becoming frozen phenomenon.

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