Monogamy property of quantum discord in photon added tripartite Glauber coherent states of GHZ-type

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Abstract

We present a complete analysis of multipartite quantum discord of photon added three-mode coherent states. We derive the explicit forms of pairwise quantum discord which characterize quantum correlations in bipartite subsystems, showing the effect of the order of the photon addition process. We also investigate the violation of the monogamy property

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1 Introduction

In the context of information processing and transmission, several theoretical and experimental results certify the potential advantages of quantum protocols compared to their classical counterparts (see for instance [1, 2, 3]). Quantum technology exploiting the intriguing phenomena of quantum world, such as entanglement, offers secure ways for communication [4, 5] and potentially powerful algorithms in quantum computation [6]. Originally, quantum information processing focused on discrete (finite-dimensional) entangled states like the polarizations of a photon or discrete levels of an atom. But, the extension from discrete to continuous variables has been proven beneficial for various kinds of quantum tasks in coding and manipulating efficiently quantum information. Coherent states, which constitute the prototypical instance of continuous-variables states, are expected to play a central role in this context. They are appealing for their mathematical elegance (continuity and over-completion property) and closeness to classical physical states of a quantum system (minimization of Heisenberg uncertainty relation). Implementing a logical qubit encoding by treating entangled coherent states as qubits in a two dimensional Hilbert space has been shown a promising strategy in performing successfully various quantum tasks such as quantum teleportation [7, 8], quantum computation [9, 10, 11], entanglement purification [12] and errors correction [13]. In view of these potential applications, a special attention was paid, during the last years, to the identification, characterization and quantification of quantum correlations in bipartite coherent states systems (see for instance the papers [14, 15, 16] and references therein). The bipartite treatment was extended to superpositions of multimode coherent states [18, 19, 20, 21, 22] which exhibit multipartite entanglement as for instance in GHZ (Greenberger-Horne-Zeilinger), W (Werner) states [23, 45] and entangled coherent state versions of cluster states [25, 26, 27]. To quantify quantum correlations beyond entanglement in coherent states systems, measures such as bipartite quantum discord [28, 29] and its geometric variant [30] were used. Explicit results were derived for quantum discord [31, 32, 33, 34, 35, 36, 37] and geometric quantum discord [38, 39, 40, 41] for some special sets of coherent states.

In other hand, decoherence is a crucial process to understand the emergence of classicality in quantum systems. It describes the inevitable degradation of quantum correlations due to experimental and environmental noise. Various decoherence models were investigated and in particular the phenomenon of entanglement sudden death was considered in a number of distinct contexts (see for instance [42] and reference therein). For optical qubits based on coherent states, the influence of the environment, is mainly due to energy loss or photon absorption. The photon loss or equivalently amplitude damping in a noisy environment can be modeled by assuming that some of field energy and information is lost after transmission through a beam splitter [36, 43]. Interestingly, it has been shown that a beam spitting device with a coherent in the first input and a number state in the second input generates photon-added coherent coherent states. In this respect, understanding the influence of adding photons process might be useful to develop the adequate strategies in improving the performance of noise reduction in quantum processing protocols. From a purely theoretical view, some some authors considered the entanglement measure in photon added multipartite coherent states [44, 45].

In this work, we derive the analytical expression of pairwise quantum discord between the three modes (parts) of the so-called quasi-GHZ coherent states. A special emphasis is devoted the process of adding photons to the first mode. Mathematically, this is represented by the action of a suitable creation operator on the the states of the first subsystem. The influence of this excitation on the quantum correlations is investigated. Another important issue in photon added GHZ-type coherent states concerns the distribution of quantum discord between the different parts of the whole system. In fact, the free shareability of quantum correlations obey an restrictive inequality termed in the literature as monogamy property [46] (see also [47, 48, 50, 51, 52]).

This paper is organized as follows. In section 2, basic definitions and equations related to photon added coherent states are presented. In particular, we consider the quantum correlations as measured by the concurrence in quasi-Bell states. In particular, we introduce the encoding mapping according with one passes from continues variables (coherent states) to discrete variables (logical quantum bits). Along the same line of reasoning, this qubit encoding is extended, in section 3, to tripartite entangled coherent states of GHZ-type. The pairwise quantum discord quantifying the amount of quantum correlations existing in the system is analytically derived. We especially study the effect of adding photon to the first mode. In section 4, we study the monogamy prperty of quantum discord. Numerical illustrations of the violation of the monogamy relation are presented for some special cases. Concluding remarks close this paper.

2 Photon added coherent states and qubit mapping

The basic objects in this work are the Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ where α is a complex number which determines the coherent amplitude of the electromagnetic field. Mathematically, a single-mode quantized radiation field is represented by the harmonic oscillator algebra spanned by the creation a^+ and annihilation a^- operators. The process of adding m photons to coherent states of type $|\alpha\rangle$ and $|-\alpha\rangle$ is usually represented by the action of the operator $(a^+)^m$ (m is a non negative integer). This process leads to generation of nonclassical states from the coherent states [62]. Several experimental as well theoretical studies were devoted to the generation and nonclassical properties of photon-added coherent states [63] (for a recent review see [64]). Explicitly, m successive actions of creation operator a^+ on the Glauber coherent states

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{1}$$

leads to the un-normalized states

$$||\alpha, m\rangle = \left(a^+\right)^m \mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sqrt{(n+m)!} \mid n+m\rangle. \tag{2}$$

The normalized m-photon added coherent states are defined by

$$|\alpha, m\rangle = \frac{(a^+)^m|\alpha\rangle}{\sqrt{\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle}}, \tag{3}$$

where

$$\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle = m!L_m(-|\alpha|^2). \tag{4}$$

In the last expression $L_m(x)$ is the Laguerre polynomial of order m defined by

$$L_m(x) = \sum_{n=0}^{m} \frac{(-1)^m m! x^n}{(n!)^2(m-n)!}. \tag{5}$$

Photon added coherent states interpolate between electromagnetic field coherent states (quasi-classical states) and Fock states $|n\rangle$ (purely quantum states). Furthermore, photon added coherent states exhibit non-classical features such as squeezing, negativity of Wigner distribution and sub Poissonian statistics. Their experimental generation using parametric down conversion in a nonlinear crystal was reported in [63]. Photon-coherent states are not orthogonal each other. Indeed using the expression

$$\langle-\alpha|(a^-)^m(a^+)^m|\alpha\rangle = e^{-2|\alpha|^2} m!L_m(|\alpha|^2), \tag{6}$$

it is simply verified that the overlapping between the states $|\alpha, m)$ and $|-\alpha, m)$ is

$$\langle-\alpha, m|\alpha, m\rangle = e^{-2|\alpha|^2} \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{7}$$

Usually coherent states are treated as continuous variable states. Recently, the idea of encoding quantum information on coherent states has led to an interesting proposal [11] in which superpositions of Glauber coherent states are used to encode logical qubits. Accordingly, one can consider an encoding scheme of type $|\alpha\rangle \longrightarrow |0\rangle$ and $|-\alpha\rangle \longrightarrow |1\rangle$ with two non orthogonal logical qubits. Alternatively, an orthogonal qubit encoding scheme involving even and odd Glauber coherent states can be be defined so that the coherent states are mapped in in $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space. Hence, for photon added coherent of type (3), one introduces the two dimensional basis spanned by two orthogonal qubits $|+, m\rangle$ and $|-, m\rangle$ defined as even and odd superpositions of photon added coherent states (3)

$$|\pm, m\rangle = \frac{1}{\sqrt{2 \pm 2\kappa_m e^{-2|\alpha|^2}}} \left(|\alpha, m) \pm |-\alpha, m)\right) \tag{8}$$

where

$$\kappa_m := \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{9}$$

Clearly, for $m = 0$, one has $\kappa_0 = 1$ and the logical qubits (8) reduce to

$$|\pm\rangle = \frac{1}{\sqrt{2 \pm 2e^{-2|\alpha|^2}}} \left(|\alpha\rangle \pm |-\alpha\rangle\right), \tag{10}$$

which coincide with even and odd Glauber coherent states providing the qubit encoding scheme introduced in [11]. It must be emphasized that such qubit encoding of paramount importance in dealing with quantum correlation in photon added coherent states and to investigate the influence of the photon adding excitation process. In this, we shall first consider the entanglement in quasi-Bell states. The quasi-Bell states are very interesting in quantum optics and serve as valuable resource for quantum teleportation and many others quantum computing operations. The quasi-Bell states

$$|\mathrm{B}_k(\alpha)\rangle = \mathcal{N}_k(\alpha)\ \big[|\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}|-\alpha\rangle \otimes |-\alpha\rangle\big]. \tag{11}$$

with $k = 0(\mod 2)$ (resp. $k = 1(\mod 2)$) stands for even (resp. odd) quasi-Bell states and the normalization factors $\mathcal{N}_k(\alpha)$ are

$$\mathcal{N}_k^{-2}(\alpha) = 2 + 2e^{-4|\alpha|^2}\cos k\pi. \tag{12}$$

By repeated actions of the creation operator on the first mode, the resulting excited quasi-Bell states given by

$$||\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha)\ \bigg[\big[(a^+)^m \otimes \mathbb{I}\big]|\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}\ \big[(a^+)^m \otimes \mathbb{I}\big]|-\alpha\rangle \otimes |-\alpha\rangle\bigg]. \tag{13}$$

are un-normalized ($\mathbb{I}$ stands for the unity operator). In fact, one verifies

$$\langle \mathrm{B}_k(\alpha, m)||\mathrm{B}_k(\alpha, m)\rangle = m!\ \frac{L_m(-|\alpha|^2) + e^{-4|\alpha|^2}L_m(|\alpha|^2)\ \cos k\pi}{1 + e^{-4|\alpha|^2}\cos k\pi}. \tag{14}$$

It is more appropriate for our purpose to deal with sates involving superpositions of normalized vectors. Thus, one introduces the normalized photon-added quasi-Bell states as

$$|\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha, m)\ \big[|m, \alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}\ |m, -\alpha\rangle \otimes |-\alpha\rangle\big], \tag{15}$$

in terms of the normalized photon added coherent state (3), with

$$\mathcal{N}_k^{-2}(\alpha, m) = 2 + 2\kappa_m e^{-4|\alpha|^2}\cos k\pi. \tag{16}$$

where κ_m is defined by (9). For $m = 0$, the normalization factor (16) reduces to (12) and the quasi-Bell states (11) are recovered. Using the qubit mapping (8) for the first mode and (10) for the second, the bipartite state (15) writes as a two qubit state

$$|\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha, m)\sum_{i=\pm}\sum_{j=\pm} C_{ij}|i, m\rangle \otimes |j\rangle \tag{17}$$

where the vectors $|i, m\rangle$ (resp. $|j\rangle$) are defined by (8) (resp. (10)) and the expansion coefficients are given by

$$C_{++} = c_m^+ c^+(1 + e^{ik\pi}), \quad C_{-+} = c^+ c_m^-(1 - e^{ik\pi}), \quad C_{+-} = c_m^+ c^-(1 - e^{ik\pi}), \quad C_{--} = c^- c_m^-(1 + e^{ik\pi}).$$

where

$$c_m^{\pm} = \sqrt{\frac{1 \pm \kappa_m e^{-2|\alpha|^2}}{2}} \qquad c^{\pm} = \sqrt{\frac{1 \pm e^{-2|\alpha|^2}}{2}}.$$

It is well established that in a pure bipartite system, the quantum discord coincides with entanglement of formation. Thus, to discuss the effect of the photon excitation of quasi-Bell states, it is sufficient to characterize the bipartite entanglement in the states (15) by means of the entanglement of formation. We recall that for ρ_{12} the density matrix for a pair of qubits 1 and 2 which may be pure or mixed, the entanglement of formation is defined by [?]

$$E(\rho_{12}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{12})|^2}) \tag{18}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and the concurrence $C(\rho_{12})$ is

$$C(\rho_{12}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{19}$$

for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ the square roots of the eigenvalues of the "spin-flipped" density matrix

$$\varrho_{12} \equiv \rho_{12}(\sigma_y \otimes \sigma_y)\rho_{12}^{\star}(\sigma_y \otimes \sigma_y). \tag{20}$$

In the last formula the star stands for complex conjugation and σ_y is the usual Pauli matrix. Thus using (19), it easy to check that the concurrence is given by

$$C_{12} = 2\mathcal{N}_k^2(\alpha, m)|C_{++}C_{--} - C_{+-}C_{-+}|, \tag{21}$$

which rewrites explicitly as

$$C_{12} = \frac{\sqrt{1 - e^{-4|\alpha|^2}}\sqrt{1 - \kappa_m^2 e^{-4|\alpha|^2}}}{1 + \kappa_m e^{-4|\alpha|^2}\cos k\pi} \tag{22}$$

in terms of the coherent states amplitude $|\alpha|$ and the excitation order m. It follows that entanglement of formation is

$$E_{12} = H\left[\frac{1}{2} + \frac{e^{-2|\alpha|^2}(1 + \kappa_m \cos k\pi)}{2 + 2\kappa_m e^{-4|\alpha|^2}\cos k\pi}\right] \tag{23}$$

For $m = 0$, one has

$$C_{12} = \frac{1 - e^{-4|\alpha|^2}}{1 + e^{-4|\alpha|^2}\cos k\pi} \tag{24}$$

In order to observe the influence of the photon excitation on the quantum entanglement of a single mode excited bipartite entangled coherent states, we need to plot the concurrence C($|\alpha|$,m) versus $|\alpha|$ for different values of the number of excited photon m. Therefore, there are two case of k (even and odd) which are shown in figure 1, 2 respectively, we can see from the figure 1, after increasing of entanglement with $|\alpha|$, it approaches as the maximum value unit when $|\alpha|$ tends the infinity. In the other hand, the entanglement degrees increases as $|\alpha|$ during a number of photon increases. We can also observe in figure 2. That C($|\alpha|$,m) increases when $|\alpha|$ increase for different values of m (m=0, 1, 2, 3, 4 and 10) respectively. Together the concurrence C($|\alpha|$,m) tends to unit for the larger $|\alpha|$.

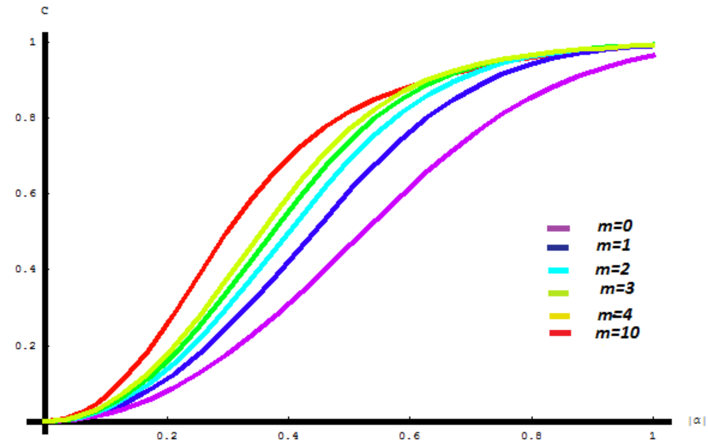

Figure 1: The concurrence C (even) of SMEECS $|\psi(|\alpha|, m)\rangle$ versus $|\alpha|$ for the different number photon excitations.

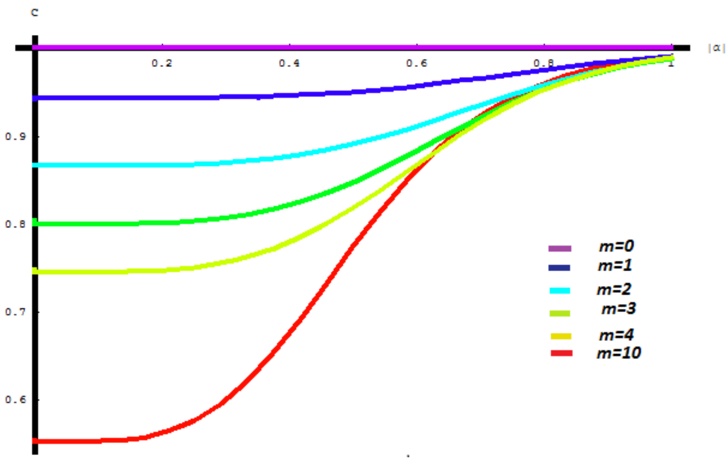

Figure 2: The concurrence C (odd) of SMEECS $|\psi(|\alpha|, m)\rangle$ versus $|\alpha|$ for the different number photon excitations.

3 Photon added quasi-GHZ coherent states

The quasi-GHZ coherent states are defined by

$$|\alpha, 0\rangle = \mathcal{N}_0(|\alpha, \alpha, \alpha\rangle + e^{ik\pi}|-\alpha, -\alpha, -\alpha\rangle). \tag{25}$$

Where the normalization constant $\mathcal{N}_0$ is given by

$$\mathcal{N}_0^{-2} = 2 + 2e^{-6|\alpha|^2}\cos k\pi, \tag{26}$$

As in the previous section, we shall consider the excitation of the first mode by adding m photon. Thus, the photon added quasi-GHZ coherent states have the form

$$|\alpha, m\rangle = \mathcal{N}a^{+m}(|\alpha, \alpha, \alpha\rangle + e^{ik\pi}|-\alpha, -\alpha, -\alpha\rangle) = \frac{\mathcal{N}}{\mathcal{N}_0}a^{+m}|\alpha, 0\rangle, \tag{27}$$

where the normalization factor is

$$\mathcal{N}^{-2} = 2m!\big[L_m(-|\alpha|^2) + e^{-6|\alpha|^2}\cos k\pi \ L_m(|\alpha|^2)\big]. \tag{28}$$

Clearly, for $m = 0$ the states $|\alpha, m\rangle$ (27) reduces to $|\alpha, m\rangle$ (25).

The tripartite state (27) can be re-equated as follows

$$|\alpha, m\rangle = \mathcal{C}_k(\alpha, m)\ (|\alpha, m) \otimes |\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}| - \alpha, m) \otimes | - \alpha\rangle \otimes | - \alpha\rangle) \tag{29}$$

so that the tree modes are described by normalized vectors. The factor $\mathcal{C}$ is defined by

$$\mathcal{C}_k^{-2}(\alpha, m) = \mathcal{N}^{-2}(\alpha, m)\big[\langle\alpha|a^{-m}a^{+m}|\alpha\rangle\big]^{-1}$$

which rewrites

$$\mathcal{C}_k^{-2}(\alpha, m) = 2 + 2e^{-6|\alpha|^2}\ \cos k\pi\ \kappa_m$$

with

$$\kappa_m = \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}.$$

In investigating the pairwise quantum discord in a tripartite system $1-2-3$, one needs the reduced density matrices describing the two qubit subsystems $1-2$, $2-3$ and $1-3$. For the states $|\alpha, m\rangle$ (29), it is simply seen that the reduced density matrices $\rho_{12} = \mathrm{Tr}_3\rho_{123}$ and $\rho_{13} = \mathrm{Tr}_2\rho_{123}$ are identical ($\rho_{123} = |\alpha, m\rangle\langle\alpha, m|$). They are given by

$$\rho_{12} = \rho_{13} = \frac{\mathcal{C}_k^2(\alpha, m)}{\mathcal{N}_k^2(\alpha, m)}\left[\left(\frac{1 + e^{-2|\alpha|^2}}{2}\right)|\mathrm{B}_k(\alpha, m)\rangle\langle\mathrm{B}_k(\alpha, m)| + \left(\frac{1 - e^{-2|\alpha|^2}}{2}\right)Z|\mathrm{B}_k(\alpha, m)\rangle\langle\mathrm{B}_k(\alpha, m)|Z\right] \tag{30}$$

in terms of photon added quasi-Bell states (15). The operator Z is the third Pauli generator defined by

$$Z|\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha, m)[|m, \alpha) \otimes |\alpha\rangle - e^{ik\pi}|m, -\alpha) \otimes | - \alpha\rangle]$$

Similarly, one obtains the reduced matrix density

$$\rho_{23} = \frac{\mathcal{C}_k^2(\alpha, m)}{\mathcal{N}_k^2(\alpha, 0)}\left[\left(\frac{1 + \kappa_m e^{-2|\alpha|^2}}{2}\right)|\mathrm{B}_k(\alpha, 0)\rangle\langle\mathrm{B}_k(\alpha, 0)| + \left(\frac{1 - \kappa_m e^{-2|\alpha|^2}}{2}\right)Z|\mathrm{B}_k(\alpha, 0)\rangle\langle\mathrm{B}_k(\alpha, 0)|Z\right] \tag{31}$$

Using the following mapping

$$|m, \pm\alpha) = \sqrt{\frac{1 + \kappa_m e^{-2|\alpha|^2}}{2}}\ |0\rangle_1 \pm \sqrt{\frac{1 - \kappa_m e^{-2|\alpha|^2}}{2}}\ |1\rangle_1 \tag{32}$$

for the first mode. For the second and third modes, have

$$| \pm \alpha\rangle = \sqrt{\frac{1 + e^{-2|\alpha|^2}}{2}}\ |0\rangle_i \pm \sqrt{\frac{1 - e^{-2|\alpha|^2}}{2}}\ |1\rangle_i \qquad i = 2, 3 \tag{33}$$

Substituting (32) and (33) in (30) (resp. (31)), one can express the density matrix ρ_{12} (resp. ρ_{23}) in the two qubit basis $\{|0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2, |1\rangle_1 \otimes |0\rangle_2, |1\rangle_1 \otimes |1\rangle_2\}$ (resp. $\{|0\rangle_2 \otimes |0\rangle_3, |0\rangle_2 \otimes |1\rangle_3, |1\rangle_2 \otimes |0\rangle_3, |1\rangle_2 \otimes |1\rangle_3\}$)

4 Quantifying the quantum discord

4.1 Bipartite measures of entanglement of formation and quantum discord

For the density matrix ρ_{12}, the total correlation is quantified by the mutual information

$$I_{12} = S_1 + S_2 - S_{12}, \tag{34}$$

where $\rho_{1(2)} = \mathrm{Tr}_{2(1)}(\rho_{12})$ is the reduced state of 1(2) and $S(\rho)$ is the von Neumann entropy of a quantum state ρ. The mutual information I_{12} contains both quantum and classical correlations. It can be decomposed as

$$I_{12} = D_{12} + C_{12}.$$

Consequently, for a bipartite quantum system, the quantum discord D_{12} is defined as the difference between total correlation I_{12} and classical correlation C_{12}. The classical part C_{12} can be determined by a local measurement optimization procedure. To remove the measurement dependence, a maximization over all possible measurements is performed and the classical correlation writes

$$C_{12} = S(\rho_2) - \widetilde{S}_{\min} \tag{35}$$

where $\widetilde{S}_{\min}$ denotes the minimal value of the conditional entropy. When optimization is taken over all measurements, the quantum discord is

$$D_{12} = I_{12} - C_{12} = S_1 + \widetilde{S}_{\min} - S_{12}. \tag{36}$$

Thus, the derivation of quantum discord requires the minimization of conditional entropy. This constitutes a complicated issue when dealing with an arbitrary mixed state. The explicit analytical expressions of quantum discord were obtained only for few exceptional two-qubit quantum states, especially ones of rank two. One may quote for instance the results obtained in [32, **?**] (see also [36, 37, 41]). The density matrix ρ_{12} is of rank two and in this case the minimization of the conditional entropy (**??**) can be performed by purifying the density matrix ρ_{12} and making use of Koashi-Winter relation [53] (see also [33]). This relation establishes the connection between the classical correlation of a bipartite state ρ_{12} and the entanglement of formation of its complement ρ_{23}. Indeed, the minimal value of the conditional entropy coincides with the entanglement of formation of ρ_{23}. It is given by

$$\widetilde{S}_{\min} = E(\rho_{23}). \tag{37}$$

It follows that the Koaschi-Winter relation and the purification procedure provide us with a computable expression of quantum discord

$$D_{12} = S_1 - S_{12} + E_{23} \tag{38}$$

when the measurement is performed on the subsystem 1.

4.2 Pairwise quantum discord

The pairwise quantum discord present in the mixed states ρ_{12}, and equivalently in ρ_{13}, can be computed using the procedure presented in the previous subsection. As result, when the measurement is performed on the subsystem $A \equiv 1$, the quantum discord is

$$D_{12} = S_1 - S_{12} + E_{23} \tag{39}$$

where k stands for the third subsystem traced out to get the reduced matrix density ρ_{ij}. The von Neumann entropy of the reduced density ρ_i is

$$S_1 = H\bigg(\frac{1}{2}\frac{(1+\kappa_m e^{-2|\alpha|^2})(1+e^{-4|\alpha|^2}\ \cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg), \tag{40}$$

and the entropy of the bipartite density ρ_{12} is explicitly given by

$$S_{12} = H\bigg(\frac{1}{2}\frac{(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)(1+e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg). \tag{41}$$

It important to emphasize that the entanglement of formation measuring the entanglement of the subsystem 2 with the ancillary qubit, required in the purification process to minimize the conditional entropy, is exactly the entanglement of formation measuring the degree of intricacy between the subsystem 2 and the traced out qubit 3. The concurrence in the subsystem $2-3$ takes the following form

$$\mathcal{C}_{23} = \kappa_m e^{-2|\alpha|^2}\ \frac{\sqrt{(1-e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}. \tag{42}$$

$$E_{23} = H\bigg(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{\kappa_m^2 e^{-4|\alpha|^2}(1-e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\bigg). \tag{43}$$

The pairwise quantum discord present in the mixed states ρ_{23} can be computed using the procedure presented in the previous subsection. As result, when the measurement is performed on the subsystem $A \equiv 2$, the quantum discord is

$$D_{23} = S_2 - S_{23} + E_{13} \tag{44}$$

where k stands for the third subsystem traced out to get the reduced matrix density ρ_{23}. The von Neumann entropy of the reduced density ρ_i is

$$S_2 = H\bigg(\frac{1}{2}\frac{(1+e^{-2|\alpha|^2})(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg), \tag{45}$$

and the entropy of the bipartite density ρ_{23} is explicitly given by

$$S_{23} = H\bigg(\frac{1}{2}\frac{(1+e^{-4|\alpha|^2}\cos k\pi)(1+\kappa_m e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg). \tag{46}$$

It important to emphasize that the entanglement of formation measuring the entanglement of the subsystem 3 with the ancillary qubit, required in the purification process to minimize the conditional entropy, is exactly the entanglement of formation measuring the degree of intricacy between the subsystem 3 and the traced out qubit 1. The concurrence in the subsystem $1-3$ takes the following form

$$\mathcal{C}_{13} = e^{-2|\alpha|^2}\ \frac{\sqrt{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}. \tag{47}$$

$$E_{13} = H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{e^{-4|\alpha|^2}(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\right). \tag{48}$$

4.3 Some special cases

- Reduced density matric ρ_{12}

The quantum discord in the state ρ_{12} is

$$\begin{aligned} D_{12} &= H\left(\frac{1}{2}\frac{(1+\kappa_m e^{-2|\alpha|^2})(1+e^{-4|\alpha|^2}\ \cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right) \\ &- H\left(\frac{1}{2}\frac{(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)(1+e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right) \\ &+ H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{\kappa_m^2 e^{-4|\alpha|^2}(1-e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\right), \end{aligned} \tag{49}$$

We start with the special case $m = 0$ (see figure 3), the state ρ_{12} is coincides with three-mode coherent states. At this point, we have $p_1 = p$ and $L_0(|\ \alpha^2\ |) = 1$, then the explicit expression of quantum discord for the density ρ_{12} is in term of the overlap p. This expression of discord is equivalent with the discord given by [**?**] for n = 3, and the discord for symmetric (antisymmetric) states m even (m odd) as shown in the figure 1. Gives a plot of quantum discord versus the overlap p for the mixed state ρ_{12}. It must be noticed that for $k = 0$, we find the maximum of discord obtained for the value p equal to 1/2. However, for $k = 1$, the quantum discord takes the maximum value for $p \to 1$ and the discord increase with p increase.

The figure 4 gives a plot of quantum discord versus the number of photon for symmetric added entangled coherent states (i.e.k even) and for different value of m. As seen from the figure, after an initial increasing, the quantum discord decreases to vanish when the amplitude of photon $|\ \alpha^2\ | \to 2$. The minimum value of discord is obtained when has no photon excitation ($i.e.m = 0$) and the maximum of quantum discord is obtained for m=10, we also found that the discord is depends on the number of the excitation photon. However, this maximum increases as m increases. In figure 5, we give a plot of the quantum discord for the antisymmetric case and for different excitation of photon number. It is remarkable that for antisymmetric quantum states the maximal value of quantum discord increases as m decreased contrarily to the symmetric states.

- Reduced density matric ρ_{23}

For the state ρ_{23}, the quantum discord is

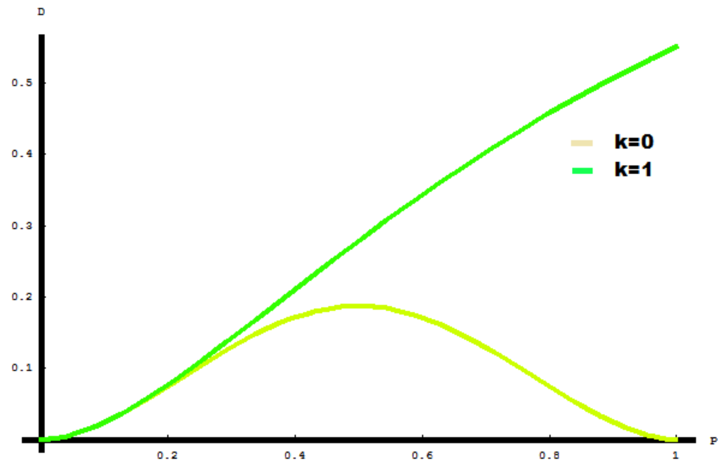

Figure 3: The quantum discord of SMEECS $|\psi(|\alpha|, m)\rangle$ versus $|\alpha|^2$ for (even) and (odd) states.

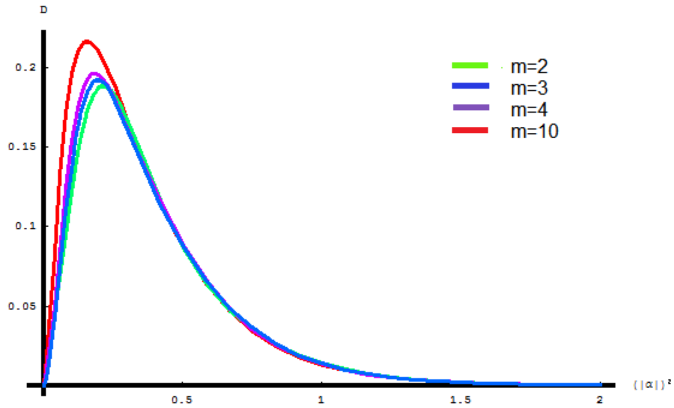

Figure 4: The quantum discord (even) of reduced density matric ρ_{12} versus $|\alpha|^2$ for the different number photon excitations.

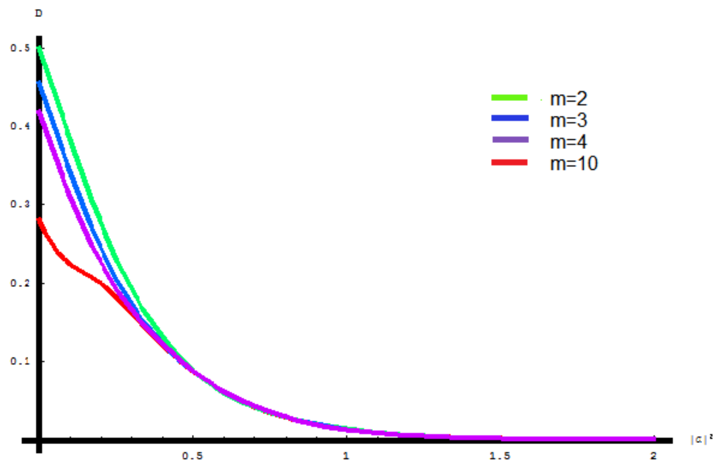

Figure 5: The quantum discord (odd) of reduced density matric ρ_{12} versus $|\alpha|^2$ for the different number photon excitations.

$$
\begin{aligned}
D_{23} &= H\left(\frac{1}{2}\frac{(1+e^{-2|\alpha|^2})(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right) \\
&- H\left(\frac{1}{2}\frac{(1+e^{-4|\alpha|^2}\cos k\pi)(1+\kappa_m e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right) \\
&+ H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{e^{-4|\alpha|^2}(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\right),
\end{aligned} \tag{50}
$$

The explicit expression of quantum discord of reduced matric ρ_{23} in the special case $m = 0$ take the same result obtained for ρ_{12}.

The behavior of quantum discord of the reduced matric ρ_{23} versus $|\alpha|^2$ for symmetric (antisymmetric case)respectively is given in the figure 6 and 7 respectively. As seen from this figures, after an initial increasing, the quantum discord decreases to vanish when $|\alpha|^2 \longrightarrow 2$. It is clearly seen that the quantum discord increases with the photon excitation number is increases in the symmetric case. Otherwise, for the antisymmetric states, the quantum discord is decreases when the photon excitation number increases.

We observed that, the minimum value ($m = 2$) of quantum discord for the reduced matric ρ_{12} is higher than the maximum value ($m = 10$) obtained for the reduced matric ρ_{23}.

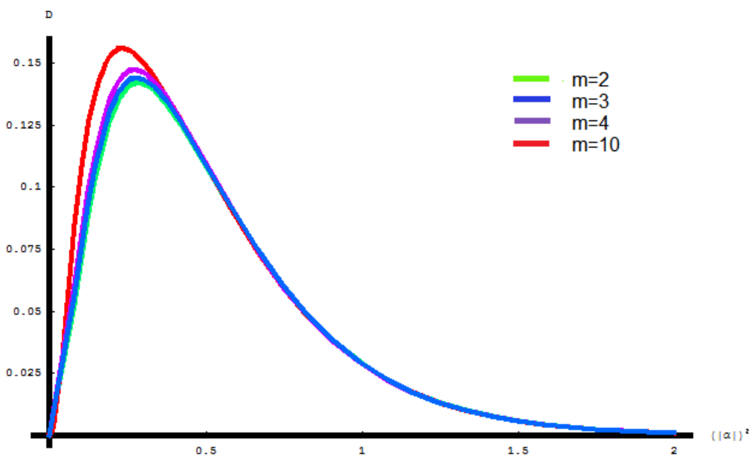

Figure 6: The quantum discord (odd) of reduced density matric ρ_{23} versus $|\alpha|^2$ for the different number photon excitations.

5 Monogamy of quantum discord for a three-qubit entangled state

Monogamy of quantum correlations is a property satisfied by certain entanglement measures in a multipartite scenario. Given a tripartite state ρ_{123}, the monogamy condition for a bipartite quantum correlation measure $\mathcal{Q}$ assures that the bipartite quantum correlations in the density operator $\rho_{1|23}$ are distributed in such a way that the following inequality is satisfied

$$\mathcal{Q}(\rho_{1|23}) \geq \mathcal{Q}(\rho_{12}) + \mathcal{Q}(\rho_{13}). \tag{51}$$

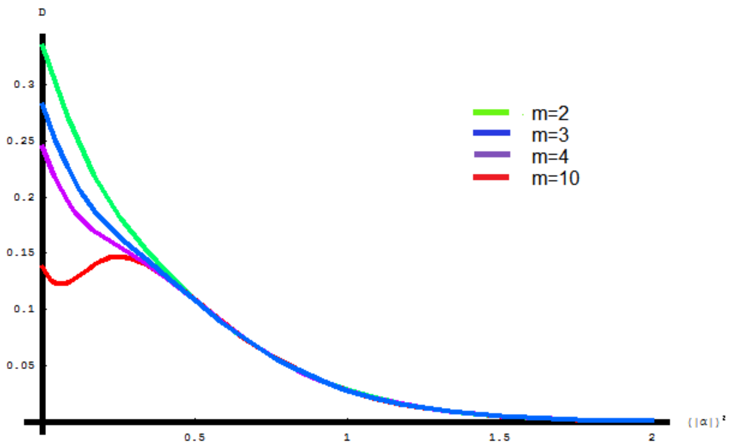

Figure 7: The quantum discord (odd) reduced density matric ρ_{23} versus $|\alpha|^2$ for the different number photon excitations.

Violation of the above inequality will imply that the quantity $\mathcal{Q}$ is polygamous for the corresponding state. Otherwise, this inequality is sufficient for quantum discord to be monogamous.
To illustrate the above analysis, we will investigate the properties of quantum discord monogamy in two different ways (quantum discord and geometric quantum discord using norm 2).

- Monogamy of quantum discord

To investigate the monogamy relation of quantum discord measured in quantum systems involving three qubits, Coffman et al [?] introduced the so called tripartite state equation (**??**). It is defined as

$$D(\rho_{1|23}) \geq D(\rho_{12}) + D(\rho_{13}), \tag{52}$$

where 1, 2 and 3 mean the respective parts of a tripartite system. Note that here $D_{1|23}$ is given by the entanglement between qubit (1) and the joint qubits (23). The quantum discord coincides with the entanglement of formation.

The pure tripartite state $|\alpha, m\rangle$ can be partitioned as bipartite system and can be expressed by means of two logical qubits. This scenario can be achieved as follows. We introduce, for the first subsystem, the orthogonal basis $\{|0\rangle_1, |1\rangle_1\}$ defined by

$$|0\rangle_1 = \frac{|\alpha, m) + |\alpha, m)}{\sqrt{2(1 + \kappa_m e^{-2|\alpha|^2})}} \qquad |1\rangle_1 = \frac{|\alpha, m) - |\alpha, m)}{\sqrt{2(1 - \kappa_m e^{-|\alpha|^2})}}. \tag{53}$$

For the modes (23), viewed as a single subsystem, we introduce the orthogonal basis $\{|0\rangle_{23}, |1\rangle_{23}\}$ given by

$$|0\rangle_{23} = \frac{|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle}{\sqrt{2(1 + e^{-4|\alpha|^2})}} \qquad |1\rangle_{23} = \frac{|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle}{\sqrt{2(1 + e^{-4|\alpha|^2})}}. \tag{54}$$

Inserting(53) and (54) in $|\alpha, m\rangle$, we get the form of the pure state $|\alpha, m\rangle$ in the basis $\{|0\rangle_1 \otimes |0\rangle_{23}, |0\rangle_1 \otimes |1\rangle_{23}, |1\rangle_1 \otimes |0\rangle_{23}, |1\rangle_1 \otimes |1\rangle_{23}\}$. Explicitly, it is given by

$$|\alpha, m\rangle = \sum_{\alpha=0,1} \sum_{\beta=0,1} C_{\alpha,\beta} |\alpha\rangle_1 \otimes |\beta\rangle_{23} \tag{55}$$

where the coefficients $C_{\alpha,\beta}$ are

$$C_{0,0} = \mathcal{C}_k(\alpha, m)(1 + e^{ik\pi})c_1^+ c_{23}^+, \qquad C_{0,1} = \mathcal{C}_k(\alpha, m)(1 - e^{ik\pi})c_1^+ c_{23}^-$$

$$C_{1,0} = \mathcal{C}_k(\alpha, m)(1 - e^{ik\pi})c_{23}^+ c_1^-, \qquad C_{1,1} = \mathcal{C}_k(\alpha, m)(1 + e^{ik\pi})c_1^- c_{23}^-.$$

in terms of the quantities

$$c_1^{\pm} = \sqrt{\frac{1 \pm \kappa_m e^{-2|\alpha|^2}}{2}} \qquad c_{23}^{\pm} = \sqrt{\frac{1 \pm e^{-4|\alpha|^2}}{2}}$$

The concurrence between the two qubits 1 and 23 is given by

$$\mathcal{C}_{1|23} = \frac{\sqrt{(1 - \kappa_m^2 e^{-4|\alpha|^2})(1 - e^{-8|\alpha|^2})}}{1 + \kappa_m e^{-6|\alpha|^2}\cos k\pi}, \tag{56}$$

from which we obtain

$$D_{1|23} = E_{1|23} = H\left(\frac{1}{2} + \frac{1}{2}\frac{\kappa_m e^{-2|\alpha|^2} + e^{-4|\alpha|^2}\cos k\pi}{1 + \kappa_m e^{-6|\alpha|^2}\cos k\pi}\right), \tag{57}$$

We will now present the conditions that signal whether a tripartite quantum state is monogamous in nature with respect to quantum discord. In figure 13, the monogamy quantity D_{123} is plotted as functions of overlapping p, for symmetric state the inequality mentioned as the above is satisfied. Otherwise, for the antisymmetric state (i.e.k=1) the monogamy relation is violated during the interval $0 \leq p \leq 1$.

In the figures 14 and 15, corresponding respectively to symmetric and antisymmetric added coherent states, we plot the monogamy quantity as a function of $|\alpha|^2$. We examine the positivity of the following inequality

$$\Delta_{123} = D_{123}(m, |\alpha|^2) = D_{1|23} - D_{12} - D_{13}, \tag{58}$$

defined in terms of the bipartite quantum discord. We shall restrict our discussion in what follows to the interval $0 \leq |\alpha|^2 \leq 1, 3$. Clearly, for symmetric states (see figure 14) the function described by (58)is non positive when $0 \leq |\alpha|^2 \leq 0.644$, consequently the quantum discord is non monogamous. Otherwise, if $|\alpha|^2 \geq 0.644$ the quantum discord is monogamous whatever the photon excitations number m. Thus, for the antisymmetric states (figure 15) the plotted curve indicates that the inequality of monogamy (58) is not satisfied if $0 \leq |\alpha|^2 \leq 0.4$, Otherwise, for $|\alpha|^2 \geq 0.4$ the quantity D_{123} is positive and the quantum discord is monogamous for everything values m.

6 Concluding remarks

Summarizing, we have presented in the early of this paper a class of the single mode excited entangled coherent states (SMEECSs)$|\psi_p(\alpha, m)\rangle$, which are obtained through actions of creation operator on the entangled coherent states. Then, we have exhibited the important properties of quantum entanglement by using different ways (specially, the concurrence, quantum discord and its version geometric). The

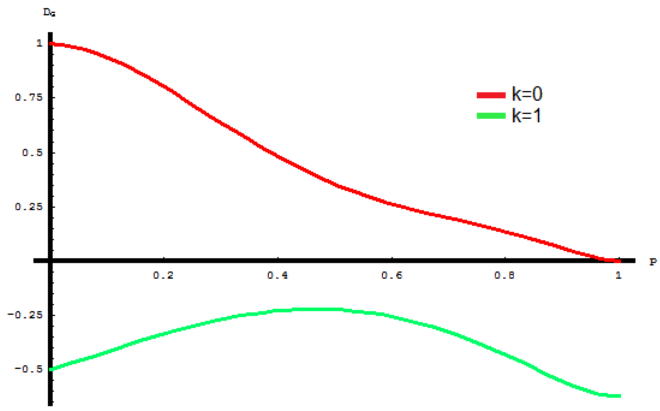

Figure 8: Monogamy D_{123} versus the overlapping p when m=0

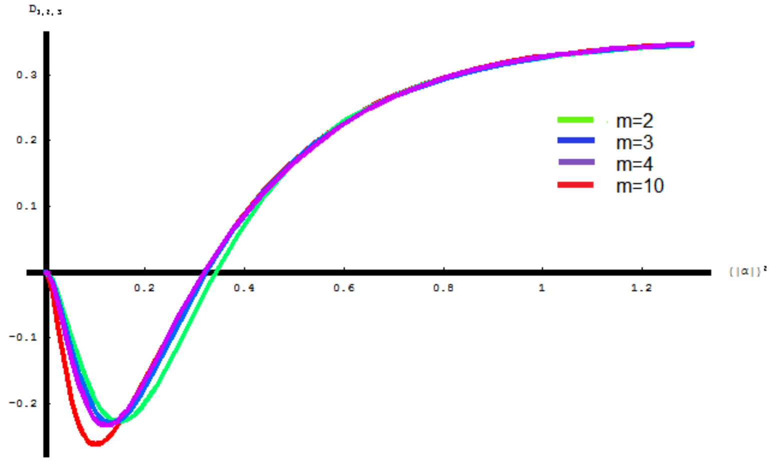

Figure 9: Monogamy D_{123} of symmetric states versus $|\alpha|^2$ for different number photon excitations.

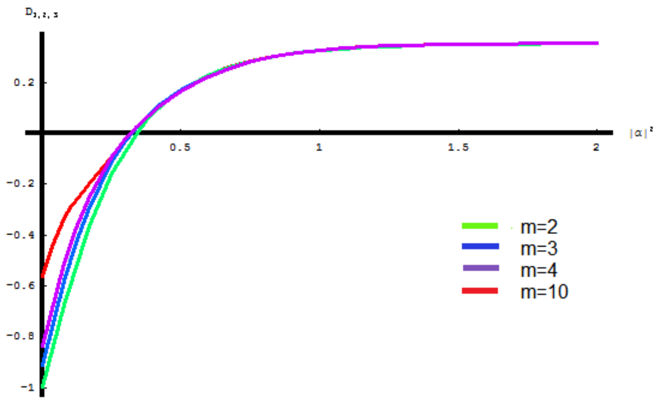

Figure 10: Monogamy D_{123} of antisymmetric states versus $|\alpha|^2$ for different number photon excitations.

first way, we have studied the concurrence for bipartite systems and investigated the influence of phonon excitations numbers on quantum entanglement. We also employed the other process for studied the quantum correlation of add coherent states for tripartite quantum states (see the equations

(**??**) and (**??**) by the quantum discord. Thus, we found two explicit analytic expressions of this measure and the results obtained are discussed. Another way which treated the quantum correlations by introducing the geometric version of quantum discord, at this stage we derived a necessary and sufficient condition. Specially, for the case of three-qubit states we have proposed in the our discussion, two version(i.e.symmetric, antisymmetric states, respectively) and the result obtained is explained in terms of different number photon excitations(i.e.the influence of m on geometric quantum discord). To close our work, we have employed the concept of quantum monogamy corresponding to quantum discord and its geometric version. In particular, we have investigated the relation between discord monogamy and a genuine tripartite entanglement measure for three-qubit pure states. Therefor, We have demonstrated that the quantum correlations examined by the entropic measure, geometric measure respectively does not satisfy the monogamy relation(51). A very important result is derived in this work, from a value determined of $|\alpha|^2$, we see that no effect of the addition of the photon can be found on the measurement of monogamy. The analysis presented in this letter can be extended to the effect of subtracting the photon of tripartite GHZ coherent states.

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