Pairwise entanglement in symmetric multi-qubit systems

Xiaoguang Wang and Klaus Mølmer
Institute of Physics and Astronomy, University of Aarhus, DK-8000, Aarhus C, Denmark.
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The concurrence, a quantitative measure of the entanglement between a pair of particles, is determined for the case where the pair is extracted from a symmetric state of N two-level systems. Examples are given for both pure and mixed states of the N-particle system, and for a pair extracted from two ensembles with correlated collective spins.

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I. INTRODUCTION

Various proposals exist for the preparation of multi-particle entangled states, and a number of these states have been pointed out to be particularly easy to prepare and to have special and useful properties. Since entanglement is defined as a property of the whole ensemble of particles, it is not immediately clear whether a sub-ensemble of particles, drawn at random from the original ensemble will also be in an entangled state, or whether the trace over some particles will destroy the quantum correlations in the system. In this paper we consider the simple questions whether a random pair of particles, extracted from a symmetric state of N two-level systems will be in an entangled state or not, where by symmetric, we assume symmetry under any permutation of the particles. Entangled states constitute a valuable resource in quantum information processing [1], and the transfer of entanglement between few qubits and the quasi-continuous variables by which we describe many-particle systems, may become an important ingredient in, e.g., quantum data-storage and inter-species teleportation.

The paper is organized as follows. In Sec. II, we present the concurrence, introduced by Wootters [2,3], who demonstrated its one-to-one correspondence with the entanglement of formation of a pair of qubits. In Sec. III, we show how the density matrix of a pair of qubits can be expressed in terms of expectation values of collective spin operators on the multi-qubit state. In Sec. IV, we analyze three examples of pure states of the N particles: spin coherent states, Dicke states, and spin squeezed states. In Sec. V, we consider an example of a mixed state with thermal entanglement [4–8], and we show examples where the pairwise entanglement depends on the temperature of the system. Finally in Sec. VI, we assume two separate ensembles in an Einstein-Podolsky-Rosen state of correlated angular momentum components, and we show that a single pair with an atom from each ensemble will be in an entangled state.

II. TWO-PARTICLE DENSITY MATRICES AND ENTANGLEMENT

It is easy to check if a pure state of two quantum systems is an entangled state or not by simply observing the eigenvalues r_i of the reduced density matrix of either system. It is also possible to quantity the amount or degree of entanglement of the state [9], $E=-\sum_i r_i \log_2 r_i$, which presents the asymptotic ratio between n and m, where n is the number of pairs in the desired state, synthesized from m pairs of maximally entangled states.

For a mixed state with density matrix ρ_{12}, a similar measure can be defined as the minium value of the weighted average of E over wave functions by which the two-particle density matrix can be written as a weighted sum. It is necessary to search for the minimum, since ρ_{12} can be written in many ways as a weighted sum of pure state projections. In the general case, this is a highly non-trivial task, as is the determination whether the state is entangled at all. For two qubits, however, entanglement is equivalent with the non-positivity of the partially transposed density matrix [10], and the entropy of formation can, magically, be obtained as a simple analytical expression [2,3]

$$E = h\left(\frac{1+\sqrt{1-\mathcal{C}^2}}{2}\right) \tag{1}$$

where $h(x)=-x\log_2 x-(1-x)\log_2(1-x)$, and where the *concurrence*, $\mathcal{C}$, is defined as

$$\mathcal{C}=\max\left\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\right\}, \tag{2}$$

where the quantities λ_i are the square roots of the eigenvalues in descending order of the matrix product

$$\varrho_{12}=\rho_{12}(\sigma_{1y}\otimes\sigma_{2y})\rho_{12}^*(\sigma_{1y}\otimes\sigma_{2y}). \tag{3}$$

In (3) ρ_{12}^* denotes the complex conjugate of ρ_{12}, and σ_{iy} are Pauli matrices for the two-level systems. The eigenvalues of ϱ_{12} are real and non-negative even though ϱ_{12} is not necessarily Hermitian, and the values of the concurrence range from zero for an unentangled state to unity for a maximally entangled state.

III. DENSITY MATRIX FOR A PAIR OF QUBITS FROM A MULTI-QUBIT STATE

A two-qubit reduced density matrix which is symmetric under exchange of the two systems can be written as

$$\rho_{12} = \begin{pmatrix} v_+ & x_+^* & x_+^* & u^* \\ x_+ & w & y^* & x_-^* \\ x_+ & y & w & x_-^* \\ u & x_- & x_- & v_- \end{pmatrix} \tag{4}$$

where the matrix elements in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ can be represented by expectation values of Pauli spin matrices of the two systems

$$\begin{aligned} v_\pm &= \frac{1}{4}\left(1 \pm 2\langle\sigma_{1z}\rangle + \langle\sigma_{1z}\sigma_{2z}\rangle\right), \\ x_\pm &= \frac{1}{2}(\langle\sigma_{1+}\rangle \pm \langle\sigma_{1+}\sigma_{2z}\rangle), \\ w &= \frac{1}{4}\left(1 - \langle\sigma_{1z}\sigma_{2z}\rangle\right), \\ y &= \langle\sigma_{1+}\sigma_{2-}\rangle, \\ u &= \frac{1}{4}(\langle\sigma_{1x}\sigma_{2x}\rangle - \langle\sigma_{1y}\sigma_{2y}\rangle + i2\langle\sigma_{1x}\sigma_{2y}\rangle). \end{aligned} \tag{5}$$

We now consider the entanglement of two qubits extracted from a symmetric multi-qubit states. If only symmetric qubit states are considered, we can describe the state of the N-qubit system in terms of the orthonormal basis $|S, M\rangle (M = -S, -S+1, ..., S)$ with $S = N/2$. The states $|S, M\rangle$ are the usual symmetric Dicke state [11], i.e., eigenstates of the collective spin operators $\vec{S}^2$ and S_z, defined as

$$S_\alpha = \frac{1}{2}\sum_{i=1}^{N} \sigma_{i\alpha}, \quad \alpha = x, y, z. \tag{6}$$

For later use it is convenient to define the number operator $\mathcal{N} = S_z + N/2$ and number states as

$$\begin{aligned} |n\rangle_N &\equiv |N/2, -N/2 + n\rangle_N, \\ \mathcal{N}|n\rangle_N &= n|n\rangle_N. \end{aligned} \tag{7}$$

The eigenvalue n of the number operator $\mathcal{N}$ is the number of qubits in the state $|0\rangle$. For example, the states $|0\rangle_N$ and $|1\rangle_N$ are explicitely written as

$$|0\rangle_N = |111...1\rangle, \tag{8}$$

$$\begin{aligned} |1\rangle_N = \frac{1}{\sqrt{N}}(&|011...1\rangle + |101..1\rangle \\ &+ ... + |111...0\rangle). \end{aligned} \tag{9}$$

$|1\rangle_N$ is also called an N–qubit W state [12,5].

Due to the symmetry of the state under exchange of particles we have

$$\begin{aligned} \langle\sigma_{1\alpha}\rangle &= \frac{2\langle S_\alpha\rangle}{N}, \\ \langle\sigma_{1+}\rangle &= \frac{\langle S_+\rangle}{N}, \\ \langle\sigma_{1\alpha}\sigma_{2\alpha}\rangle &= \frac{4\langle S_\alpha^2\rangle - N}{N(N-1)}, \\ \langle\sigma_{1x}\sigma_{2y}\rangle &= \frac{2\langle[S_x, S_y]_+\rangle}{N(N-1)}, \\ \langle\sigma_{1+}\sigma_{2z}\rangle &= \frac{\langle[S_+, S_z]_+\rangle}{N(N-1)}, \end{aligned} \tag{10}$$

where $[A, B]_+ = AB + BA$ is the anticommutator for operators A and B.

From Eqs.(5) and (10) we may thus express the density matrix elements of ρ_{12} in terms of the expectation values of the collective operators,

$$\begin{aligned} v_\pm &= \frac{N^2 - 2N + 4\langle S_z^2\rangle \pm 4\langle S_z\rangle(N-1)}{4N(N-1)}, \\ x_\pm &= \frac{(N-1)\langle S_+\rangle \pm \langle[S_+, S_z]_+\rangle}{2N(N-1)}, \\ w &= \frac{N^2 - 4\langle S_z^2\rangle}{4N(N-1)}, \\ y &= \frac{2\langle S_x^2 + S_y^2\rangle - N}{2N(N-1)}, \\ u &= \frac{\langle S_x^2 - S_y^2\rangle + i\langle[S_x, S_y]_+\rangle}{N(N-1)} = \frac{\langle S_+^2\rangle}{N(N-1)}. \end{aligned} \tag{11}$$

IV. PURE MULTIQUBIT STATES

In this section we study three examples, where the N two-level systems are described by a pure state which is invariant under permutation of the particles.

A. Spin coherent states

The spin coherent state [13] is obtained by a rotation of the spin state $|S, M = S\rangle$, which in turn is the product state of all N particles in the $|0\rangle$ state. Hence it is a separable state. It is still interesting to go through the above procedure and to insert the explicit expression of the spin coherent state [13],

$$|\eta\rangle = (1 + |\eta|^2)^{-N/2}\sum_{n=0}^{N}\begin{pmatrix} N \\ n \end{pmatrix}^{1/2}\eta^n|n\rangle_N, \tag{12}$$

where η is chosen real in the following. By a straightforward calculation from Eqs.(11) and (12), we find

$$\rho_{12} = \frac{1}{(1+\eta^2)^2}\begin{pmatrix} \eta^4 & \eta^3 & \eta^3 & \eta^2 \\ \eta^3 & \eta^2 & \eta^2 & \eta \\ \eta^3 & \eta^2 & \eta^2 & \eta \\ \eta^2 & \eta & \eta & 1 \end{pmatrix} \tag{13}$$

which is in agreement with our observation that the two-particle state is really a product state of two rotated spin-$\frac{1}{2}$ particles in the states $(\eta|0\rangle + |1\rangle)/\sqrt{1+\eta^2}$. The matrix product ϱ_{12} is found to be a 4×4 matrix of zero's, revealing the role of the σ_y Pauli matrices in (3): ρ_{12} is the projection operator on spin states with a definite direction in the xz-plane, the application of σ_y is equivalent to a 180° rotation in the xz-plane, and ϱ_{12} is therefore the vanishing product of projection operators on two orthogonal subspaces. Naturally, the concurrence vanishes in this case, $\mathcal{C} = 0$; there is no pairwise entanglement in the spin coherent state.

B. Dicke State $|N/2, M\rangle$

The Dicke states, defined as effective number states above, are states with a definite number of particles occupying the internal states $|0\rangle$ and $|1\rangle$. Such states may in principle be prepared in an atomic physics experiment by Quantum Non-Demolition detection of the atomic populations by phase contrast imaging of the atomic sample [14,15]. By rotation of all spins, a separable spin coherent state is first prepared with a binomial distribution on the various Dicke states, cf., Eq.(12), and experiments have already demonstrated a factor 3 reduction in the variance of the populations after such a detection [16].

From Eq.(11), it is easy to see that the reduced density matrix ρ_{12} is given by

$$\rho_{12} = \begin{pmatrix} v_+ & 0 & 0 & 0 \\ 0 & w & w & 0 \\ 0 & w & w & 0 \\ 0 & 0 & 0 & v_- \end{pmatrix} \tag{14}$$

with matrix elements

$$v_\pm = \frac{(N \pm 2M)(N - 2 \pm 2M)}{4N(N-1)},$$
$$w = \frac{N^2 - 4M^2}{4N(N-1)}. \tag{15}$$

The concurrence of a simple density matrix of the form (38) with $x_\pm, u = 0$, and $y = y^*$ is given by [17]

$$\mathcal{C} = 2 \max\{0, y - \sqrt{v_+ v_-}\}, \tag{16}$$

where we have used that $2y + v_+ + v_- = 1$ to ρ_{12}. Now substituting Eq.(15) into (16), we explicitly obtain

$$\mathcal{C} = \frac{1}{2N(N-1)}\{N^2 - 4M^2 - \sqrt{(N^2 - 4M^2)[(N-2)^2 - 4M^2]}\}. \tag{17}$$

The values of $\mathcal{C}$ for different N and M are illustrated in Fig. 1. For any Dicke state except the ones with maximum $|M|$, if one extracts two particles, they will be in an entangled state. We also observe that the concurrence is nearly a constant in the neighborhood of $M = 0$.

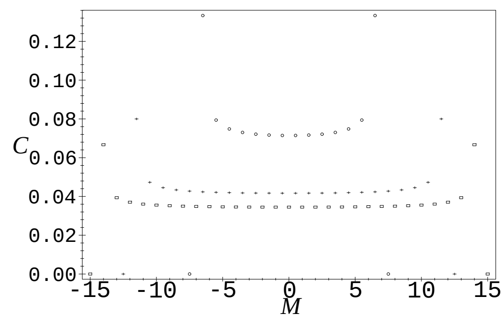

FIG. 1. The concurrence in the Dicke state for different number N. $N = 15$(open circles), $N = 25$ (crosses), and $N = 30$ (open square).

The variation of M around $M = 0$ is small for an initial binomial distribution with this mean value, and the concurrence will be very close to the exact result, $\mathcal{C} = 1/(N-1)$ for $M = 0$, irrespective of the outcome of a QND measurement of M.

The Dicke states $|N/2, M = \pm(N/2 - 1)\rangle$ have a concurrence of $\mathcal{C} = 2/N$. These states are identical with the W state (see Eq.(9)), which are known to be the symmetric states with the highest possible concurrence [18].

C. Kitagawa-Ueda state

In 1993, Kitagawa and Ueda proposed a nonlinear Hamiltonian χS_x^2 in order to generate spin squeezed states [19]. This effective Hamiltonian may be realized in ion traps [20], where it was already implemented in order to produce multi-particle entangled states (of four particles) [21], and it may be implemented in two-component Bose-Einstein condensates as a direct consequence of the collisional interactions between the particles [22], see also [23].

When the Hamiltonian $H = \chi S_x^2$ is applied to the many-particle system, which has been prepared in the product state $|0\rangle_N = |111, ..., 1\rangle$, the wave function at time t is obtained as

$$|\Psi(t)\rangle = e^{-i\chi t S_x^2}|0\rangle_N. \tag{18}$$

Using the results obtained in [19] the following expectation values are obtained ($\mu = 2\chi t$)

$$\langle S_x \rangle = \langle S_y \rangle = 0,$$
$$\langle S_z \rangle = -\frac{N}{2}\cos^{N-1}\left(\frac{\mu}{2}\right)$$
$$\langle S_x^2 \rangle = N/4$$

$$\langle S_y^2 \rangle = \frac{1}{8}\left(N^2 + N - N(N-1)\cos^{N-2}\mu\right)$$
$$\langle S_z^2 \rangle = \frac{1}{8}\left(N^2 + N + N(N-1)\cos^{N-2}\mu\right)$$
$$\langle [S_+, S_z]_+ \rangle = 0$$
$$\langle [S_x, S_y]_+ \rangle = \frac{1}{2}N(N-1)\cos^{N-2}\frac{\mu}{2}\sin\frac{\mu}{2}. \quad (19)$$

We are now able to determine the two-particle density matrix, which is on the form

$$\rho_{12} = \begin{pmatrix} v_+ & 0 & 0 & u^* \\ 0 & w & w & 0 \\ 0 & w & w & 0 \\ u & 0 & 0 & v_- \end{pmatrix} \quad (20)$$

with matrix elements given by Eq.(11). The combination of Eqs.(11) and (19) gives explicitly the matrix elements.

From Eqs.(2) and (3), the concurrence for the matrix (20) is obtained as

$$\mathcal{C} = \begin{cases} 2\max(0, |u| - w), & \text{if } 2w < \sqrt{v_+ v_-} + |u|; \\ 2\max(0, w - \sqrt{v_+ v_-}), & \text{if } 2w \geq \sqrt{v_+ v_-} + |u|. \end{cases} \quad (21)$$

The concurrence of the spin squeezed states is given by analytical expressions in the argument $\mu = 2\chi t$, which are too lengthy to present her. In Fig. 2 we present the results numerically: If two atoms are extracted at random from spin squeezed samples they will be in a mutually entangled state. We observe that the concurrence is symmetric with respect to $\mu = \pi$. At this special point of $\mu = \pi$ the N–particle GHZ state is produced [20], and it has no pairwise entanglement.

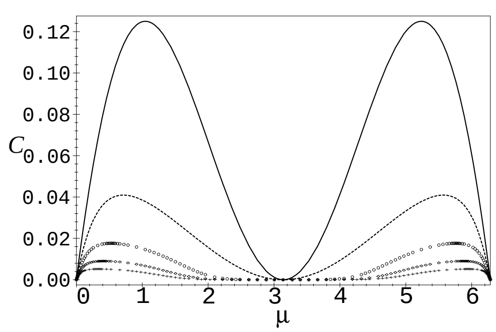

FIG. 2. The concurrence as a function of μ for different number N. $N = 3$(solid line), $N = 4$ (dashed line), $N = 5$ (open circles), $N = 6$ (open diamonds), and $N = 7$ (crosses).

V. MIXED MULTIQUBIT STATES AND THERMAL ENTANGLEMENT

An interesting and novel type of thermal entanglement was introduced and analyzed within the Heisenberg XXX [4], XX [5], and XXZ [6] models as well as within the Ising model in a magnetic field [7]. The state of the system at thermal equilibrium is represented by the density operator $\rho(T) = \exp\left(-H/kT\right)/Z$, where $Z = \text{tr}[\exp\left(-H/kT\right)]$ is the partition function, H the system Hamiltonian, k is Boltzmann's constant which we henceforth take equal to unity, and T the temperature. As $\rho(T)$ represents a thermal state, the entanglement in the state is called *thermal entanglement* [4]. Unlike in standard statistical physics where all properties are obtained from the partition function, determined by the eigenvalues of the system, entanglement properties require in addition knowledge of the eigenstates. The analytical results in the previous studies on thermal entanglement are only available for two [4–7] and three qubits [8]. Here we consider pairwise entanglement in the multiqubit systems.

A. Isotropic Heisenberg model

We consider the N–qubit isotropic Heisenberg Hamiltonian

$$H_I = \frac{J}{4}\sum_{i \neq j}^{N}\left(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z\right) \quad (22)$$

The positive (negative) J corresponds to the antiferromagnetic (ferromagnetic) case. In this model all particles interact with each other.

By using the collective spin operators, the Hamiltonian is rewritten as

$$H = J\left(S_x^2 + S_y^2 + S_z^2\right) = J\vec{S}^2 \quad (23)$$

up to a trivial constant.

Unlike pure states, the symmetric multi-particle density matrix does not only populate the fully symmetric Dicke states, and we have to determine the number of collective spin-S states for each S. Write S as $(N/2 - k)$, we know that for $k = 0$, a single irreducible representation exists: the $N + 1$ fully symmetric Dicke states with $S = N/2$. The number of irreducible representation with $S = N/2 - 1$ is obtained by noting that their maximum M value is also $N/2 - 1$, and a total of $\begin{pmatrix} N \\ 1 \end{pmatrix} = N$ states exist with precisely one particle in the $|1\rangle$ state. One of these belong to the $S = N/2$ irreducible representation, and the remaining $N - 1$ states must have $S = N/2 - 1$. This argument can now be repeated to obtain the number of states with $S = N/2 - 2$ and $M = N/2 - 2$, i.e., the number of $S = N/2 - 2$ irreducible representation, etc., until all 2^N states of the system have been accounted for.

The isotropic Hamiltonian only depends on $\vec{S}^2$, and knowing the multiplicity of each value of this quantity we write the partition function

$$Z = \sum_{k=0}^{N/2} N_k [2(N/2-k)+1] e^{-\beta J (N/2-k)(N/2-k+1)}, \quad (24)$$

where $N_k = \begin{pmatrix} N \\ k \end{pmatrix} - \begin{pmatrix} N \\ k-1 \end{pmatrix}$ follows from the above argument. We assume $\begin{pmatrix} N \\ -1 \end{pmatrix} = 0$.

The reduced density matrix for two qubits is

$$\rho_{12} = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & y & 0 \\ 0 & y & w & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \quad (25)$$

with matrix elements given by Eq.(11). The matrix element $v = v_\pm$ since $\langle S_z \rangle = 0$.

Due to the symmetry property $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle$, we only need to determine

$$\langle S_z^2 \rangle = \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} (m - N/2 + k)^2 \\ e^{-\beta J (N/2-k)(N/2-k+1)}/Z. \quad (26)$$

And from Eq.(16), the concurrence is obtained as

$$\begin{aligned} \mathcal{C} &= \frac{1}{2N(N-1)} \max\{0, \\ &\quad 2|2\langle S_x^2 + S_y^2 \rangle - N| - N^2 + 2N - 4\langle S_z^2 \rangle\} \\ &= \frac{1}{2N(N-1)} \max\{0, \\ &\quad 2|4\langle S_z^2 \rangle - N| - N^2 + 2N - 4\langle S_z^2 \rangle\}. \quad (27) \end{aligned}$$

To identify the sign of $A \equiv 2|4\langle S_z^2 \rangle - N| - N^2 + 2N - 4\langle S_z^2 \rangle$ in (27), we consider the case where $4\langle S_z^2 \rangle \geq N$, for which $A = 4\langle S_z^2 \rangle - N^2$. Since $\langle S_z^2 \rangle \leq \frac{1}{3}\frac{N}{2}\left(\frac{N}{2}+1\right) < \frac{N^2}{4}$, we always have $4\langle S_z^2 \rangle - N^2 < 0$, and there is no pairwise entanglement. In the opposite case where $4\langle S_z^2 \rangle < N$, we have $A = 4N - 12\langle S_z^2 \rangle - N^2$, since $\langle S_z^2 \rangle \geq 0$, we have $A \leq 0$ if $N \geq 4$. For case of $N = 3$, we have shown that the pairwise thermal entanglement is absent from both the antiferromagnetic and ferromagnetic isotropic model [8].

So we conclude that there is no thermal entanglement for $N \geq 3$ in the isotropic Heisenberg model. The case of $N = 2$ is discussed in detail in Ref. [4] and it is shown that there is no thermal entanglement for the ferromagnetic case. In order to observe the pairwise entanglement in the multiqubit system, now we consider the anisotropic Heisenberg model.

B. Anisotropic Heisenberg model

The anisotropic Heisenberg Hamiltonian is given by

$$H_a = J\left(S_x^2 + S_y^2 + \Delta S_z^2\right) = J\vec{S}^2 + J(\Delta - 1)S_z^2, \quad (28)$$

where Δ is the anisotropy parameter. Obviously the Hamiltonian H_a reduces to H_I when $\Delta = 1$, and H_a yields the XX model when $\Delta = 0$.

The concurrence is still given by (27), but the partition function and the relevant expectation values now become

$$\begin{aligned} Z &= \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} e^{-\beta J(\Delta-1)(m-N/2+k)^2} \quad (29) \\ &\quad \times e^{-\beta J(N/2-k)(N/2-k+1)}, \\ \langle S_z^2 \rangle &= \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} (m - N/2 + k)^2 \\ &\quad \times e^{-\beta J(\Delta-1)(m-N/2+k)^2} \\ &\quad \times e^{-\beta J(N/2-k)(N/2-k+1)}/Z, \quad (30) \\ \langle S_x^2 + S_y^2 \rangle &= \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} [(N/2-k)(N/2-k+1) \\ &\quad -(m-N/2+k)^2] \\ &\quad \times e^{-\beta J(\Delta-1)(m-N/2+k)^2} \\ &\quad \times e^{-\beta J(N/2-k)(N/2-k+1)}/Z. \quad (31) \end{aligned}$$

This model leads to pairwise entanglement, as shown by the numerical results presented in Figure 3 as functions of the reciprocal temperature, $x = \beta J$. For $N = 2$ we observe that the concurrence is symmetric with respect to $x = 0$, which is consistent with the result in Ref. [5]. In other words, the thermal entanglement appears for both the antiferromagnetic and ferromagnetic cases. However for $N \geq 3$, the thermal entanglement only exists for the ferromagnetic case. We observe a critical value of x, after which the entanglement vanishes. And the critical value increases as N increases.

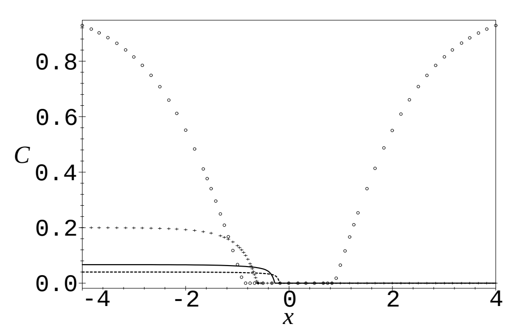

FIG. 3. The concurrence as a function of $x = \beta J$ for different number N in the XX model ($\Delta = 0$): $N = 2$(open circle), $N = 5$ (crosses), $N = 15$ (solid line), and $N = 25$ (dashed line).

Within the above framework we may also consider more general models such as

$$H_g = J\vec{S}^2 + f(S_z), \tag{32}$$

where $f(S_z)$ is an arbitrary analytical function of S_z. As the operator $f(S_z)$ commutes with $\vec{S}^2$, similar analytical results for the concurrence can be obtained and the thermal entanglement can be readily generated for special choices of $f(S_z)$.

VI. EPR-CORRELATED ENSEMBLES

Finally we consider two EPR-correlated ensembles. This state is not invariant under any permutation of particles, but only under those permutations that exchange particles within each ensemble, and it is furthermore characterized by the correlations between the samples 1 and 2:

$$(J_{1x} - J_{2x})|\Psi\rangle = 0, \tag{33}$$
$$(J_{1y} + J_{2y})|\Psi\rangle = 0. \tag{34}$$

A state that obeys Eqs.(33) and (34) can in principle be obtained by successive QND detection of the observables $J_{1x} - J_{2x}$ and $J_{1y} + J_{2y}$ [24,25]. Equivalently the above equations can be written as

$$(J_{1+} - J_{2-})|\Psi\rangle = 0, \tag{35}$$
$$(J_{1-} - J_{2+})|\Psi\rangle = 0. \tag{36}$$

It is easy to check that a solution of the above equation is the EPR-correlated state

$$|\Psi\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |n\rangle_N \otimes |n\rangle_N \tag{37}$$

And it also satisfies $(J_{1z} - J_{2z})|\Psi\rangle = 0$.

Now we consider the entanglement of two qubits, which belong to different ensembles. we first identify the two-qubit reduced density matrix :

$$\rho_{12} = \begin{pmatrix} v & 0 & 0 & u^* \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ u & 0 & 0 & v \end{pmatrix} \tag{38}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, which can be represented by

$$\begin{aligned} v &= \frac{1}{4}\left(1 \pm 2\langle\sigma_{1z}\rangle + \langle\sigma_{1z}\sigma_{2z}\rangle\right), \\ w &= \frac{1}{4}\left(1 - \langle\sigma_{1z}\sigma_{2z}\rangle\right) = \frac{1}{4} - \frac{\langle J_{1z}J_{2z}\rangle}{N^2}, \\ u &= \frac{1}{4}(\langle\sigma_{1x}\sigma_{2x}\rangle - \langle\sigma_{1y}\sigma_{2y}\rangle + i2\langle\sigma_{1x}\sigma_{2y}\rangle) \\ &= \frac{\langle J_{1+}J_{2+}\rangle}{N^2}. \end{aligned} \tag{39}$$

The concurrence is given by

$$\begin{aligned} \mathcal{C} &= 2\max\{0, |u| - w\} \\ &= 2\max\{0, \frac{\langle J_{1+}J_{2+}\rangle + \langle J_{1z}J_{2z}\rangle}{N^2} - \frac{1}{4}\} \end{aligned} \tag{40}$$

The expectation values of $J_{1+}J_{2+}$ and $J_{1z}J_{2z}$ are readily obtained in the state $|\Psi\rangle$, and we find that $\mathcal{C} = 1/N$. The pair of particles is in an entangled state. If the ensembles are really macroscopic, as in [25], the entanglement is, however, very weak.

VII. CONCLUSIONS

The purpose of this paper has been to point out that multi-particle entanglement quite typically implies pairwise entanglement within the sample. We showed that the two-particle density matrix is readily expressed in terms of expectation values of collective operators, in the case of symmetrical states of the many-particle system, and we provided the value of the concurrence for a number of examples. These results confirmed and generalized results obtained, e.g., on the pairwise entanglement in systems with definite ($N = 3, 4$) numbers of particles.

The entropy of formation, and the very issue of entanglement, is highly non-trivial for situations dealing with more than two particles, and for mixed states of systems with dimensions higher than 2. Studying and optimizing the two-particle concurrence in systems with many particles may be a useful way to learn about the more complicated case.

ACKNOWLEDGMENTS

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