## Abstract

This paper deals with a Schwinger realization of polynomial $su(2)$ algebras. It involves generalized Weyl-Heisenberg algebras $A_\kappa(1)$ ($\kappa$ characterizes the deviation from the usual boson algebra). Hilberitian as well as analytical representations are defined. To show the interest for such algebras, we investigate the dynamical symmetry of the Kepler system in a two-dimensional curved space and we discuss their relevance in the super-symmetric quantum mechanics.

## I. INTRODUCTION

The generalized $su(2)$ algebras developed in the context of the theory of quantum algebras have attracted a lot of interest in the literature. This is especially by motivated their relevance in different areas of physics as for instance exactly solvable quantum potentials, statistical physics, Yang-Mills theory, field theory, two dimensional integrable models, quantum optics, etc. [1-17]. Many variants of generalized algebras were investigated for different purposes. The generalized $su(2)$ algebras can be defined in a unified scheme. Indeed, n terms of the raising and lowering generators $J_\pm$ and the diagonal one $J_3$, the very well-known linear $su(2)$ algebra is characterized by the structure relations

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = f(J_3) \tag{1}$$

where the structure function characterizes the deviation from the linear $su(2)$ algebra. Clearly, many different generalization schemes can be defined. of particular interest are polynomial $su(2)$ alegebras in which $f(J_3)$ is a polynomial in $J_3$. For such algebras, the Casimir operator is given by

$$C = J_+J_- + g(J_3 - 1) = J_-J_+ + g(J_3) \tag{2}$$

with the function $g(J_3)$ satisfies

$$g(J_3) - g(J_3 - 1) = f(J_3) \tag{3}$$

Unitary irreducible representations were for some specific forms of the polynomial structure function $f(J_3)$ (see for instance [14]). Coherent states associated with specific polynomial $su(2)$ algebras were defined [23-26]. Also bosonic realizations of such algebras by means of generalized bosons were discussed in the literature [23-26]. They can classified into two main categories. One, essentially based on Jackson's $q$- calculus, uses the concept of $q$-deformed boson algebra ($q \in C$) firstly proposed by Arik and Coon [28] and developed further by Macfarlane [29] and Biedenharn [30] to provide bosonic and fermionic realizations of quantum algebras. The second category involves new variants of generalized Weyl-Heisenberg algebras [31-33]. The main characteristics of the these algebras is the fact that are not related to ones defined in the framework of quantum algebras and the $q$-calculus. Among them, we shall consider the single boson extended Weyl-Heisenberg algebra $\mathcal{A}_\kappa(1)$ [31]. It is generated by the operators $a^+$, $a^-$ and $N$ satisfying the following commutation relations

$$[a^-, a^+] = 1 + 2\kappa N, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \tag{4}$$

labeled by a real parameter $\kappa$. The usual boson algebra is recovered when $\kappa$ goes to zero. In this context, the main of this work is as follows. We shall introduce a one parameter family of nonlinear $su(2)$ algebras of type (1). This is realized à la Schwinger in terms of two commuting copies of generalized $\mathcal{A}_\kappa(1)$ algebras (4). The representation spaces are defined using the Fock representation of $\mathcal{A}_\kappa(1)$ algebras derived in [31]. To provide physical illustration of such polynomial generalized bosonic realizations, we give in the second section the general procedure to derive the Casimir operator polynomial $su(2)$ algebras of an arbitrary order. Section III is devoted to the Schwinger realization of polynomial $su(2)$ algebras and the associated discrete representation spaces. In section IV, Fock-Bragmann representations and coherent states are constructed. To exemplify the relevance of this family of nonlinear algebras, some physical models are discussed in Section V. Closing remarks close this paper.

## II. TWO $\mathcal{A}_\kappa$-BOSON REALIZATION OF $su(2)$ ALGEBRA

The two generalized boson construction discussed in this section is a straightforward generalization of the usual Schwinger realization. We consider two commuting copies of the generalized Weyl-Heisenberg algebra $\mathcal{A}_\kappa(1)$ (4). Each one is spanned by the three linear operators $a_i^-$, $a_i^+$ and $N_i$ ($i = 1, 2$) satisfying the following relations

$$[a_i^-, a_i^+] = I + 2\kappa N_i \qquad [N_i, a_i^\pm] = \pm a_i^\pm \qquad \left(a_i^-\right)^\dagger = a_i^+ \qquad N_i^\dagger = N_i. \tag{5}$$

We denote the Fock space finite or infinite dimensional

$$\mathcal{F}_\kappa = \mathcal{F}_1 \otimes \mathcal{F}_2 = \{|n_1, n_2\rangle, \ n_1, n_2 \ \text{ ranging}\}. \tag{6}$$

The actions of creation, annihilation and number operators on $\mathcal{F}$ are defined by

$$\begin{aligned} a_i^+|n_1, n_2\rangle &= \sqrt{F(n_i+1)}\ |n_1 + s_i^-, n_2 + s_i^+\rangle, \\ a_i^-|n_1, n_2\rangle &= \sqrt{F(n_i)}\ |n_1 - s_i^-, n_2 - s_i^+\rangle, \\ a_1^-|0, n_2\rangle = 0 \quad & a_2^-|n_1, 0\rangle = 0 \quad N_i|n_i\rangle = n_i\ |n_i\rangle \end{aligned} \tag{7}$$

where the quantity $s_i^\pm$ is defined by

$$s_i^\pm = \frac{1}{2}(1 \pm (-)^i).$$

In (), the positive function is given by [31]

$$F(n_i) = n_i[1 + \kappa(n_i - 1)] \qquad i = 1, 2. \tag{8}$$

It is quadratic in $n_i$ except the special case $\kappa = 0$ where it reduces to a linear function. The Hilbertian representation of $\mathcal{A}_\kappa(1)$ was investigated in [31]. Indeed, it has been shown that for $\kappa \geq 0$, the Fock space $\mathcal{F}$ is infinite dimensional. It follows that for two generalized bosons of type (4), the Fock state is

$$\mathcal{F}_{\kappa\geq 0} \equiv \mathcal{F}_+ = \{|n_1, n_2\rangle,\ n_1 \in \mathbf{N},\ n_2 \in \mathbf{N}\}.$$

For $\kappa < 0$, the algebra $\mathcal{A}_\kappa(1)$ admits finite dimensional Hilbert space [31]. Indeed, it is simply verified from (8) that for $\kappa < 0$, the function $F(n_i)$ is positive for the integers $n_i \leq d - 1$ where $d$ stands for the integer part of $\frac{-1}{\kappa}$. Thus, in this case, the Fock space is finite dimensional

$$\mathcal{F}_{\kappa<0} \equiv \mathcal{F}_- = \{|n_1, n_2\rangle,\ n_1 = 0, 1, \cdots, d-1; \quad n_2 = 0, 1, \cdots, d-1\},$$

and the dimension is $d^2$.

As we are interested in nonlinear extension of $su(2)$ using a Schwinger procedure based on the generalized Weyl-Heisenberg algebra $\mathcal{A}_\kappa(1)$, we decompose the Fock spaces in angular momentum subspaces. In this respect, we introduce the quantum numbers $j$ and $m$ defined as

$$j = \frac{1}{2}(n_1 + n_2) \qquad m = \frac{1}{2}(n_1 - n_2)$$

and we set the following correspondence

$$|n_1, n_2\rangle \equiv |\frac{n_1 + n_2}{2}, \frac{n_1 - n_2}{2}\rangle = |j, m\rangle$$

It follows that one can write the Fock spaces $\mathcal{F}_+$ and $\mathcal{F}_-$ as

$$\mathcal{F}_+ = \bigoplus_{j=0}^{\infty} \mathcal{F}_j = \bigoplus_{j=0}^{\infty} \{|j, m\rangle,\ m = -j, -j+1, \cdots, j-1, +j\}$$

and

$$\mathcal{F}_- = \bigoplus_{j=0}^{d} \mathcal{F}_j = \bigoplus_{j=0}^{d-1} \{|j, m\rangle,\ m = -j, -j+1, \cdots, j-1, +j\}$$

Clearly, for $\kappa < 0$, the Schwinger realization produces only angular momentum with $j = 0, \frac{1}{2}, 1, \cdots d-1$ such that $\dim\mathcal{F}_- = d^2$. This constitutes the main difference with the case where $\kappa$ is positive including the case $\kappa = 0$ which gives the standard Schwinger realization of $su(2)$ algebra where all possible values of $j$ can be generated. In the Schwinger picture, the $su(2)$ operators are realized as

$$J_3 = \frac{1}{2}(N_2 - N_1), \quad J_0 = \frac{1}{2}(N_2 + N_1)$$

$$J_+ = a_2^\dagger a_1, \quad J_- = a_2 a_1^\dagger, \tag{9}$$

This gives

$$[J_+, J_-] = 2J_3\Big(1 - \kappa + 2\kappa J_0(1 + \kappa J_0)\Big) + 4\kappa^2 J_3^2. \tag{10}$$

and the Casimir operator

$$C = 2J_- J_+ + 2J_3(J_3 + 1)\Big(1 - \kappa + 2\kappa J_0(1 + \kappa J_0)\Big) + 2\kappa^2 J_3^2 (J_3 + 1)^2 \tag{11}$$

To simplify the notations in deriving the coherent states in the following subsection a suitable basis for the subspaces $\mathcal{F}_j$ is introduced as follows. We set

$$|j, m\rangle = |n\rangle \quad \text{with} \quad n = j + m.$$

In this basis, the actions of the generalized $su(2)$ algebra rewrites

$$J_-|n\rangle = \sqrt{f(n)}|n-1\rangle$$

$$J_-|n\rangle = \sqrt{f(n+1)}|n+1\rangle$$

$$J_3|n\rangle = (n - j)|n+1\rangle$$

where

$$f(n) = F(n)F(2j - n + 1)$$

in terms of the structure function of the generalized Weyl-Heisenberg algebra $\mathcal{A}_\kappa(1)$ given by (8). Notice that the function $f(N)$ is of order 4 in $N$ and the obtained non linear $su(2)$ algebra can be idenified with polynomial Weyl-Heisenberg algebra of order 4 introduced in [32].

## III. BARGMANN REALIZATION AND COHERENT STATES

According to Bargmann prescription, the coherent states provides the analytical realizations of a given algebra. For the nonlinear algebra $su(2)$ discussed above, for a representation

$j$, the coherent states are of the form

$$|z\rangle = \sum_{n=0}^{2j} a_n z^n |n\rangle \tag{12}$$

where $z$ is a complex variable, $n = j + m$ and the $a_n$ coefficients to be determined. In the Bargmann realization any vector is realized as follows

$$|n\rangle \longrightarrow a_n z^n \equiv \langle \bar{z}|n\rangle \tag{13}$$

and the operator $J^-$ is assumed acts as a derivation according to

$$J_- \longrightarrow \frac{d}{dz}. \tag{14}$$

Thus, using the expression of the action of $J_-$ on the Fock space and the Bargamnn correspondence, it is simple to see that the coefficients $a_n$ satisfy the recursion relation

$$n a_n = \sqrt{f(n)} a_{n-1} \tag{15}$$

where

$$f(n) = F(2j - n + 1)F(n).$$

This yields

$$a_n = a_0 \frac{\sqrt{f(n)!}}{n!} \tag{16}$$

where $f(n)! = f(1)f(2)\cdots f(n)$ with $f(0)! = 1$. The coefficient $a_o$ is obtained from the normalization condition of the coherent states . As result, one gets

$$|z\rangle = \mathcal{N}^{-1} \sum_{n=0}^{2j} \frac{\sqrt{f(n)!}}{n!} z^n |n\rangle \tag{17}$$

where the $\mathcal{N}$ normalization factor is given by the sum

$$|\mathcal{N}|^2 = \sum_{n=0}^{2j} \frac{f(n)!}{(n!)^2} |z|^{2n}. \tag{18}$$

In this realization, the operators $J_3$ and $J_+$ act, respectively, as follows

$$N \longrightarrow z\frac{d}{dz} - j \qquad J_+ \longrightarrow z\Big(1 + \kappa z\frac{d}{dz}\Big)\Big(2j - z\frac{d}{dz}\Big)\Big(1 + 2j\kappa - \kappa z\frac{d}{dz}\Big). \tag{19}$$

Any state $|\Psi\rangle$

$$|\Psi\rangle = \sum_{n=0}^{2j} \Psi_n |n\rangle$$

corresponds to an analytical function given by the following correspondence

$$|\Psi\rangle \longrightarrow \Psi(z) = \mathcal{N}\langle \bar{z}|\Psi\rangle = \sum_{n=0}^{2j} \Psi_n f_n(z) \tag{20}$$

where the monomials $f_n(z)$ are the analytical functions associated with the vectors $n$

$$f_n(z) = \mathcal{N}\langle \bar{z}|2n\rangle = \frac{\sqrt{f(n)!}}{n!} z^n. \tag{21}$$

It easy to check

$$J_3 f_n(z) = (j-n) f_n(z)$$
$$J_+ f_n(z) = \sqrt{f(n+1)} f_{n+1}(z)$$
$$J_- f_n(z) = \sqrt{f(n)} f_{n-1}(z)$$

It is simple to verify that, in the limiting case $\kappa \to 0$, one recovers the $su(2)$ coherent states and the standard Bargamnn realization based on spin coherent states.

### A. The $\mathcal{A}_{\{\kappa\}}$ algebra

We start with the generalized Weyl-Heisenberg algebra on $C$ spanned by the linear operators $a^-$ (annihilation operator), $a^+$ (creation operator) and $N$ (number operator) satisfying the commutation relations

$$[a^-, a^+] = G(N) \qquad [N, a^-] = -a^- \qquad [N, a^+] = +a^+ \tag{22}$$

and the Hermitian conjugation conditions

$$a^+ = (a^-)^\dagger \qquad N = N^\dagger. \tag{23}$$

The $G$ function in (22) is such that

$$G(N) = (G(N))^\dagger. \tag{24}$$

Of course, the case $G(N) = I$, where $I$ is the identity operator, corresponds to the usual Weyl-Heisenberg algebra or harmonic oscillator algebra. Various realizations of $G$ are known in the literature [**? ? ? ? ? ? ? ? ? ?** ]. In the present paper, we shall be concerned with a class of polynomial Weyl-Heisenberg algebras characterized by

$$G(N) = F(N+I) - F(N) \tag{25}$$

with the $F$ function defined by

$$F(N) = N[I + \kappa_1(N - I)][I + \kappa_2(N - I)] \cdots [I + \kappa_r(N - I)] \tag{26}$$

where the $\kappa_i$'s ($i = 1, 2, \cdots, r$) are real parameters (for instance, see [? ]). We note $\mathcal{A}_{\{\kappa\}}$, with $\{\kappa\} \equiv \{\kappa_1, \kappa_2, \cdots, \kappa_r\}$, the generalized Weyl-Heisenberg algebra (or generalized oscillator algebra) defined via (22)-(26).

The $F(N)$ polynomial of order $r + 1$ with respect to $N$ can be developed as

$$F(N) = N \sum_{i=0}^{r} s_i (N - I)^i \tag{27}$$

in terms of the coefficients (totally symmetric under permutation group $S_r$)

$$s_0 = 1 \qquad s_i = \sum_{j_1 < j_2 < \cdots < j_i} \kappa_{j_1} \kappa_{j_2} \cdots \kappa_{j_i} \quad (i = 1, 2, \cdots, r) \tag{28}$$

where the indices $j_1, j_2, \cdots, j_i$ take the values $1, 2, \cdots, r$. Then, the $G(N)$ operator can be written

$$G(N) = I + \sum_{i=1}^{r} s_i \Big[ (N + I)N^i - N(N - I)^i \Big] \tag{29}$$

which clearly indicates that $\mathcal{A}_{\{\kappa\}}$ with $\{\kappa\} \equiv \{0, 0, \cdots, 0\}$ coincides with the usual Weyl-Heisenberg algebra.

The $\mathcal{A}_{\{\kappa\}}$ $r$-parameter algebra covers the cases of (i) the extended harmonic oscillator algebra [? ], (ii) the fractional oscillator algebra [? ], and (iii) the $W_k$ algebra introduced in the context of fractional supersymmetric quantum mechanics of order $k$ [? ? ]. As a particular case, algebra $\mathcal{A}_{\{\kappa\}}$ with $\kappa_1 = \kappa$ and $r = 1$ is nothing but the $\mathcal{A}_\kappa$ algebra worked out in [? ] and corresponding to

$$G(N) = I + 2\kappa N. \tag{30}$$

Algebra $\mathcal{A}_\kappa$ defined by (22), (23) and (30) turns out to be of particular interest when dealing with dynamical symmetries of some exactly solvable quantum systems. More precisely, $\mathcal{A}_{\kappa=0}$ corresponds to the usual oscillator system while $\mathcal{A}_{\kappa<0}$ and $\mathcal{A}_{\kappa>0}$ are relevant to the Morse and Pöschl-Teller systems, respectively [? ? ]. Note also that the $\mathcal{A}_\kappa$ one-parameter algebra provides a unified scheme to deal with the $su(2)$ algebra (for $\kappa < 0$), the $su(1,1)$ algebra (for $\kappa > 0$), and the usual Weyl-Heisenberg algebra (for $\kappa = 0$) [? ? ]. More generally, the $\mathcal{A}_{\{\kappa\}}$ algebra can be viewed as a special class of the polynomial extensions of $su(2)$ and $su(1,1)$ discussed in [? ] and [? ], respectively.

## IV. CONCLUDING REMARKS

In this paper, we discussed the realization of polynomial $su(2)$ algebra by generalizing the standard Schwinger realization. In this generalized realization, we used two generalized boson characterized by the extended Weyl-Heisenberg algebra $\mathcal{A}_\kappa(1)$ [31]. Other kinds of extended algebras were recently introduced in the literature and in this respect, we believe that the procedure developed in this work can be easily adapted to generate other variants of polynomial $su(2)$ algebras. In the same spirit, the Bargmann realization discussed here may be naturally applied to construct the associated coherent states.

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