xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

xxxx[a1] and **yyyy**[b,c 2]

[a]*Department of Physics , Faculty of Sciences, University Ibnou Zohr, Agadir , Morocco*

[b]*LPHE-Modeling and Simulation, Faculty of Sciences, University Mohammed V, Rabat, Morocco*

[c]*Centre of Physics and Mathematics, CPM, CNESTEN, Rabat, Morocco*

Abstract

We study the evolution of geometric quantum discord (GMQD) of a two qubits system coupled with two independent bosonic reservoirs. We consider sub-ohmic, ohmic and super-ohmic. A special attention is devoled to Dicke states and their superpositions.

[1]email: xxxx
[2]email: yyyy

1 Introduction

Entanglement has been attracting much attention from physicists in both theory and experiment [1][2][3][4][5], as it plays an important role in quantum information processing such as quantum communications [2] and quantum cryptography [6]. However, there are some exceptions. For instance, quantum computation [7] based on one pure qubit does not employ entanglement, but it needs other non-classical resources. This suggests that entanglement is not the unique resource but just one kind of quantum correlation. In fact, there is another kind of quantum correlation named quantum discord.

Quantum discord, as a measure of bipartite non-classical correlation, is a promising candidate and has generated a lot of interest. For a quantum state ρ in a composite Hilbert space $H = H_A \otimes H_B$, the total amount of correlation can be quantified by quantum mutual information [8]:

$$I(\rho_{AB}) = H(\rho_A) + H(\rho_B) - H(\rho_{AB}) \tag{1}$$

with $H(\rho) = -Tr[\rho \log_2 \rho]$ the von Neumann entropy and $\rho_{A(B)} = Tr_{B(A)}\rho$ the reduced density matrix by tracing system $B(A)$.

Quantum discord is a measure of nonclassical correlations that may include entanglement but is an independent measure. We will document with simple examples that the amounts of classical correlation, quantum discord and entanglement bear no simple relationship to each other. Taking system A as the apparatus, the quantum discord is defined as follows:

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) \tag{2}$$

where $C(\rho_{AB})$ denotes the classical correlations of the state.

The optimization procedure involved in the calculation of quantum discord prevents one to write an analytical expression for quantum discord even for simple two-qubit systems. Quantum discord is analytically computed only for a few families of states including the Bell-diagonal states [9][10], two-qubit X states [11][12], two-qubit rank-2 states [13], a class of rank-2 states of $4 \otimes 2$ systems [14], and Gaussian states of the continuous variable systems [15]. Moreover, based on the optimization of the conditional entropy, an algorithm to calculate the quantum discord of the two-qubit states is presented in . It is also important to have some computable bounds on the quantum discord and some authors have obtained such bounds [16],[17].

This paper is organized as follows. In section 2, we give general expression of GMQD for two qubits in Bosonic Reservoirs and discuss the GMQD for a kind of X-state. In section 3, the Geometric quantum discord. We study in section 4, the GMQD for Reservoirs with Ohmic-Like spectral densities specially the GMQD for Dicke states and their superpositions (generalized W states and GHZ states, the superpositions of two Dicke states). In section 4, the conclusion.

2 Two spin in independent bosonic reservoirs

2.1 Bipartite states

Here, we consider an ensemble of N spin-1/2 particles described by the collective angular momentum operators

$$J_\alpha = \frac{1}{2}\sum_{k=1}^{N}\sigma_{k\alpha}, \quad \alpha = x, y, z. \tag{3}$$

Since the QD is adopted to describe the nonlocal quantum correlation, it is necessary to derive the pairwise density matrix. In this paper, we only study the pairwise correlations. For all collective spin models, including the Dicke model and the LMG model, the pairwise reduced density matrix in the standard basis, $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$ (with $\sigma_z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$) [?], can be derived as

$$\rho = \begin{pmatrix} v_+ & x_+^* & x_+^* & u^* \\ x_+ & w & y & x_-^* \\ x_+ & y & w & x_-^* \\ u & x_- & x_- & v_- \end{pmatrix}. \tag{4}$$

The detailed expressions for these elements are

$$\begin{aligned} v_\pm &= \frac{N^2 - 2N + 4\langle J_z^2\rangle \pm 4(N-1)\langle J_z\rangle}{4N(N-1)}, \\ x_\pm &= \frac{(N-1)\langle J_+\rangle \pm \langle [J_+, J_z]_+\rangle}{2N(N-1)}, \\ w &= \frac{N^2 - 4\langle J_z^2\rangle}{4N(N-1)}, y = \frac{\langle J_x^2 + J_y^2\rangle - N/2}{N(N-1)}, \\ u &= \frac{\langle J_+^2\rangle}{N(N-1)}, \end{aligned} \tag{5}$$

where $[A, B]_+ = AB + BA$. $w = y$, for $\sum_{\alpha=x,y,z} J_\alpha^2 = J^2 = \frac{N}{2}(\frac{N}{2}+1)$. In particular, since the Dicke and LMG models have symmetric ground states with parity conservation, we find $x_\pm = 0$ [?, ?]. Hence the pairwise reduced density matrix in X form is shown as

$$\rho = \begin{pmatrix} v_+ & 0 & 0 & u^* \\ 0 & w & y & 0 \\ 0 & y & w & 0 \\ u & 0 & 0 & v_- \end{pmatrix}. \tag{6}$$

For the two-qubit states in X form, the QD may be derived analytically, according to Ref. [?].

2.2 Coupling to bosonic reservoirs

The model under consideration is a quantum system of two qubits coupled to two independant bosonic reservoirs, the Hamiltonian of wich may be expressed as

$$H = \sum_{j=1}^{2}[\frac{\nu_j}{2}\sigma_{j,3} + \sum_k \omega_{j,k} b_{j,k}^\dagger b_{j,k} + \sum_k \sigma_{j,3}(g_{j,k} b_{j,k}^\dagger + g_{j,k}^* b_{j,k}], \tag{7}$$

where ν_j is the energy difference between the excited state $|1\rangle_j$ and the ground state $|0\rangle_j$, and $\sigma_{j,3}$ is the Pauli matrix of qubit j with $\sigma_{j,3}|1\rangle_j = |1\rangle_j$ and $\sigma_{j,3}|0\rangle_j = -|0\rangle_j$. $b^\dagger_{j,k}(b_{j,k})$ and $\omega_{j,k}$ denote the bosonic creation (annihilation) operator and the frequency of the kth mode of the reservoir of the qubit j,respectively.$g_{j,k}$ denotes the coupling strength between the qubit j and the kth mode. The Hamiltonian describes the spin-boson model without tunneling. A possible experiment setup is a double-dot charge qubit placed in a freestanding semiconductor slab.

Suppose that the two qubits are initially writes in the X states,

$$\rho_s(0) = \frac{1}{4}(I \otimes I + x_3\sigma_3 \otimes I + y_3 I \otimes \sigma_3 + \sum_{i,j=1}^{3} R_{ij}\sigma_i \otimes \sigma_j), \tag{8}$$

the reservoirs are in thermal equilibrium states at temperature T,

$$\rho_{E_j} = exp(-\beta \sum_k \omega_{j,k} b^\dagger_{j,k} b_{j,k})/Z_{E_j}, \tag{9}$$

and the whole system is in the product state,

$$\rho(0) = \rho_s(0) \otimes \rho_{E_1} \otimes \rho_{E_2} \tag{10}$$

where c_i is a real number with $0 \leq |c_i| \leq 1$ for each i,Z_{E_j} is the partition function of the reservoir j with $Z_{E_j} = Tr(exp(-\beta \sum_k \omega_{j,k} b^\dagger_{j,k} b_{j,k})) = \prod_k (1 - e^{-\beta\omega_{j,k}})^{-1}$, and $\beta = 1/(k_B T)$. The whole system?s state at time t is governed by

$$\rho(t) = exp(-iHt)\rho(0)exp(iHt), \tag{11}$$

and the state of the two qubits at time t can be obtained by the partial trace

$$\rho_s(t) = Tr_E[exp(-iHt)\rho(0)exp(iHt)]. \tag{12}$$

In order to calculate GMQD, we need to write $\rho_s(t)$ in the form of Eq [**?**]. From Eq [**?**], the elements of $\rho_s(t)$ can be expressed as

$$\langle mm'|\, \rho_s(t) \,|nn'\rangle = Tr[\xi_1^{mn}(t)\xi_1^{m'n'}(t)\rho_(0)]. \tag{13}$$

where $\xi_1^{mn}(t) = e^{iHt}\xi_1^{mn}e^{-iHt}$ is the Heisenberg operator of qubit j, and $\xi_1^{mn} = |n\rangle_j \langle m|$. Noting that $[\xi_1^{mn}, H] = 0$, we have $\xi_1^{mn}(t) = \xi_1^{mn}$. The operators $\xi_1^{mn}(t)$ for $m \neq n$ can be obtained by solving the Heisenberg equations of motion,

$$i\frac{d}{dt}b_{j,k} = \omega_{j,k}b_{j,k}(t) + g_{j,k}\sigma_{j,3}, \tag{14}$$

$$i\frac{d}{dt}\xi_j^{01}(t) = -\nu_j\xi_j^{01}(t) - 2\sum_k [g_{j,k}b^\dagger_{j,k}(t) + g^*_{j,k}b_{j,k}(t)]\xi_j^{01}(t) \tag{15}$$

The solutions to the above differential equations are

$$b_{j,k}(t) = e^{-i\omega_{j,k}t}[b_{j,k} + \frac{1}{2}\alpha_{j,k}(t)\sigma_{j,3}] \tag{16}$$

$$\xi_j^{01}(t) = \xi_j^{01} exp\left\{ i\nu_j t - \sum_k [\alpha_{j,k}(t) b_{j,k}^\dagger - \alpha_{j,k}(t) b_{j,k}] \right\}, \tag{17}$$

with

$$\alpha_{j,k} = 2g_{j,k}(1 - e^{i\omega_{j,k}(t)})/\omega_{j,k}. \tag{18}$$

with the help of Eqs.(8) and (12),we finally obtain

$$\rho_s(t) = \frac{1}{4}(I\otimes I + T_{30}\sigma_3\otimes I + T_{03} I\otimes\sigma_3 + T_{11}\sigma_1\otimes\sigma_1 + T_{12}\sigma_1\otimes\sigma_2 + T_{21}\sigma_2\otimes\sigma_1 + T_{22}\sigma_2\otimes\sigma_2 + T_{33}\sigma_3\otimes\sigma_3), \tag{19}$$

Where

$$T_{11} = [R_{11}cos(\nu_1 t)cos(\nu_2 t) + R_{22}sin(\nu_1 t)sin(\nu_2 t) - R_{12}cos(\nu_1 t)sin(\nu_2 t) - R_{21}sin(\nu_1 t)cos(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{12} = [R_{11}cos(\nu_1 t)sin(\nu_2 t) - R_{22}sin(\nu_1 t)cos(\nu_2 t) + R_{12}cos(\nu_1 t)cos(\nu_2 t) - R_{21}sin(\nu_1 t)sin(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{21} = [R_{11}sin(\nu_1 t)cos(\nu_2 t) - R_{22}cos(\nu_1 t)sin(\nu_2 t) - R_{12}sin(\nu_1 t)sin(\nu_2 t) + R_{21}cos(\nu_1 t)cos(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{22} = [R_{11}sin(\nu_1 t)sin(\nu_2 t) + R_{22}cos(\nu_1 t)cos(\nu_2 t) + R_{12}sin(\nu_1 t)cos(\nu_2 t) + R_{21}cos(\nu_1 t)sin(\nu_2 t)]e^{-\gamma(t)}$$

$$T_{30} = x_3$$

$$T_{03} = y_3$$

$$T_{33} = R_{33} \tag{20}$$

and

$$\gamma(t) = \sum_{j=1}^{2}\sum_k 4\left|g_{j,k}\right|^2 \omega_{j,k}^{-2} coth(\frac{\beta\omega_{j,k}}{2})[1 - cos(\omega_{j,k}t)]. \tag{21}$$

$$\rho_s(t) = \begin{pmatrix} \rho_{00}(t) & 0 & 0 & \rho_{03}(t) \\ 0 & \rho_{11}(t) & \rho_{12}(t) & 0 \\ 0 & \rho_{21}(t) & \rho_{22}(t) & 0 \\ \rho_{30}(t) & 0 & 0 & \rho_{33}(t) \end{pmatrix} \tag{22}$$

where

$$\rho_{00}(t) = 1 + T_{30} + T_{03} + T_{33} = \rho_{00}$$

$$\rho_{03}(t) = T_{11} - T_{22} - iT_{12} - iT_{21} = e^{-\gamma(t)}\rho_{03}[cos(\nu_1 t + \nu_2 t)) - isin(\nu_1 t + \nu_2 t)]$$

$$\rho_{11}(t) = 1 + T_{30} - T_{03} - T_{33} = \rho_{11}$$

$$\rho_{12}(t) = T_{11} + T_{22} + iT_{12} - iT_{21} = e^{-\gamma(t)}\rho_{12}[cos(\nu_1 t - \nu_2 t) - isin(\nu_1 t - \nu_2 t)]$$

$$\rho_{21}(t) = T_{11} + T_{22} - iT_{12} + iT_{21} = e^{-\gamma(t)}\rho_{21}[cos(\nu_1 t - \nu_2 t) + isin(\nu_1 t - \nu_2 t)]$$

$$\rho_{22}(t) = 1 - T_{30} + T_{03} - T_{33} = \rho_{22}$$

$$\rho_{30}(t) = T_{11} - T_{22} + iT_{12} + iT_{21} = e^{-\gamma(t)}\rho_{30}[cos(\nu_1 t + \nu_2 t) + isin(\nu_1 t + \nu_2 t)]$$

$$\rho_{33}(t) = 1 - T_{30} - T_{03} + T_{33} = \rho_{33} \tag{23}$$

These matrix elements can be written also as follows

$$\rho_{00}(t) = \rho_{00}$$

$$\rho_{03}(t) = e^{-\gamma(t)}e^{-i(\nu_1+\nu_2)t}\rho_{03}$$

$$\rho_{11}(t) = \rho_{11}$$

$$\rho_{12}(t) = e^{-\gamma(t)}e^{-i(\nu_1-\nu_2)t}\rho_{12}$$

$$\rho_{21}(t) = e^{-\gamma(t)}e^{i(\nu_1-\nu_2)t}\rho_{21}$$

$$\rho_{22}(t) = \rho_{22}$$

$$\rho_{30}(t) = e^{-\gamma(t)}e^{i(\nu_1+\nu_2)t}\rho_{30}$$

$$\rho_{33}(t) = \rho_{33} \tag{24}$$

2.3 Closest classical states to two qubit X states

To begin, we shall present the procedure leading to the closest classically correlated state to the two-qubit X state (??). The Fano-Bloch representation (??) reads

$$\rho_{12} = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + T_{03}\sigma_0 \otimes \sigma_3 + T_{30}\sigma_3 \otimes \sigma_0 + \sum_{kl} T_{kl}\sigma_k \otimes \sigma_l\right] \tag{25}$$

where the correlation matrix elements are obtainable from (??) modulo some obvious substitutions. The geometric measure of quantum discord is defined as the distance the state ρ_{12} and its closest classical-quantum state presenting zero discord [?]

$$D_{\rm g}(\rho_{12}) = \min_{\chi_{12}} ||\rho_{12} - \chi_{12}||^2 \tag{26}$$

where the Hilbert-Schmidt norm is defined by $||X||^2 = {\rm Tr}(X^\dagger X)$ and the minimization is taken over the set of all classical states. When the measurement is performed on the qubit 1, the classical states write

$$\chi_{12} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^2 + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^2 \tag{27}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit 1, p_i $(i = 1, 2)$ stands for probability distribution and ρ_i^2 $(i = 1, 2)$ is the marginal density of the qubit 2. The classically correlated states χ_{12} can also be written as

$$\chi_{12} = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} te_i \quad \sigma_i \otimes \sigma_0 + \sum_{i=1}^{3}(s_+)_i \ \sigma_0 \otimes \sigma_i + \sum_{i,j=1}^{3} e_i(s_-)_j \ \sigma_i \otimes \sigma_j\right] \tag{28}$$

where

$$t = p_1 - p_2, \qquad e_i = \langle \psi_1 | \sigma_i | \psi_1 \rangle, \qquad (s_\pm)_j = \mathrm{Tr}\big((p_1 \rho_1^2 \pm p_2 \rho_2^2)\sigma_j\big).$$

It follows that the distance between the density matrix ρ_{12} and the classical state χ_{12}, as measured by Hilbert-Schmidt norm, is then given by

$$||\rho_{12} - \chi_{12}||^2 = \frac{1}{4}\bigg[(t^2 - 2te_3T_{30} + T_{30}^2) + \sum_{i=1}^{3}(T_{0i} - (s_+)_i)^2 + \sum_{i,j=1}^{3}(T_{ij} - e_i(s_-)_j)^2\bigg] \tag{29}$$

The minimization of the distance (29), with respect to the parameters t, $(s_+)_i$ and $(s_-)_i$, gives

$$t = e_3 T_{30}$$

$$(s_+)_1 = 0 \quad (s_+)_2 = 0 \quad (s_+)_3 = T_{03}$$

$$(s_-)_i = \sum_{j=1}^{3} e_j T_{ji}. \tag{30}$$

Inserting these solutions in (29), one has

$$||\rho_{12} - \chi_{12}||^2 = \frac{1}{4}\bigg[\mathrm{Tr}K - \vec{e}^{\,t} K \vec{e}\bigg] \tag{31}$$

where the matrix K is defined by

$$K = xx^\dagger + TT^\dagger \tag{32}$$

with

$$x^\dagger = (0, 0, T_{30}) \qquad T = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}.$$

From equation (31), one see that the minimal value of Hilbert-Schmidt distance (31) is reached for the largest eigenvalue of the matrix K. We denote by λ_1, λ_2 and λ_3 the eigenvalues of the matrix K (32) corresponding to the X state (**??**) or equivalently (25). They are given by

$$\lambda_1 = 4(|\rho_{14}| + |\rho_{23}|)^2, \qquad \lambda_2 = 4(|\rho_{14}| - |\rho_{23}|)^2, \qquad \lambda_3 = 2[(\rho_{11} - \rho_{33})^2 + (\rho_{22} - \rho_{44})^2]. \tag{33}$$

To get the minimal value of the Hilbert-Schmidt distance (31) and subsequently the amount of geometric quantum discord, one compares λ_1, λ_2 and λ_3. As λ_1 is always greater than λ_2, the largest eigenvalue $\lambda_{\max}$ is λ_1 or λ_3. It follows that the geometric discord is given by

$$D_g(\rho_{12}) = \frac{1}{4}\min\{\lambda_1 + \lambda_2, \lambda_2 + \lambda_3\}. \tag{34}$$

To write down the explicit expressions of the closest classical state χ_{12} to ρ_{12}, one has to determine the eigenvector $\vec{e}_{\max}$ associated with the largest eigenvalue $\lambda_{\max}$. In this respect, two cases ($\lambda_{\max} = \lambda_1$ and $\lambda_{\max} = \lambda_3$) are separately discussed. We begin by density matrices ρ_{12} (**??**) whose entries satisfy the condition $\lambda_{\max} = \lambda_3$. The associated eigenvector is given by $\vec{e}_3 = (0,0,1)$. Replacing in the set of constraints (30), one has

$$\chi_{12}^3 = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + T_{30} \ \ \sigma_3 \otimes \sigma_0 + T_{03} \ \sigma_0 \otimes \sigma_3 + T_{33} \ \sigma_3 \otimes \sigma_3\right] \tag{35}$$

In the second situation, the eigenvector corresponding to λ_1 is given by $\vec{e}_1 = (\cos\frac{\phi}{2}, -\sin\frac{\phi}{2}, 0)$ where $e^{i\phi} = \frac{\rho_{14}\rho_{23}}{|\rho_{14}||\rho_{23}|}$. Reporting the components of $\vec{e}_1$ in (30), one gets the closest classical state

$$\chi_{12}^1 = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + T_{30} \ \ \sigma_3 \otimes \sigma_0 + \sum_{i=1}^{2}\sum_{j=1}^{2} \tilde{T}_{ij} \ \sigma_i \otimes \sigma_j\right] \tag{36}$$

where

$$\tilde{T}_{11} = \cos\frac{\phi}{2}(\cos\frac{\phi}{2}T_{11} - \sin\frac{\phi}{2}T_{21}) \qquad \tilde{T}_{12} = \cos\frac{\phi}{2}(\cos\frac{\phi}{2}T_{12} - \sin\frac{\phi}{2}T_{22})$$
$$\tilde{T}_{21} = -\sin\frac{\phi}{2}(\cos\frac{\phi}{2}T_{11} - \sin\frac{\phi}{2}T_{21}) \qquad \tilde{T}_{22} = -\sin\frac{\phi}{2}(\cos\frac{\phi}{2}T_{12} - \sin\frac{\phi}{2}T_{22}).$$

2.4 Evolution of geometric quantum discord

3 Geometric measure of quantum discord under quantum decoherence channels

A quantum channel can be described in the Kraus representation

$$\mathcal{E}(\rho) = \sum_{\mu} K_\mu \rho K_\mu^\dagger, \tag{37}$$

where K_μ are Kraus operators satisfying $\sum_\mu K_\mu^\dagger K_\mu = \sigma_0$. As we discussed in the previous section, to obtain the GMQD, we need to know the expectation values of the Pauli matrices of the two qubits for the state $\mathcal{E}(\rho)$. So we turn to the Heisenberg picture to describe quantum channels via the map [**?**]

$$\mathcal{E}^\dagger(A) = \sum_{\mu} K_\mu^\dagger A K_\mu \tag{38}$$

with A an arbitrary observable. Then the expectation value of A can be obtained through $\langle A\rangle = \mathrm{Tr}\left[A\mathcal{E}(\rho)\right] = \mathrm{Tr}\left[\mathcal{E}^\dagger(A)\rho\right]$. Because an arbitrary Hermitian operator on $\mathbb{C}^2$ can be expressed by $A = \sum_{i=0}^{3} r_i\sigma_i$ with $r_i \in \mathbb{R}$, then a quantum channel for a qubit can be characterized by the transmission matrix M defined through

$$\mathcal{E}^\dagger(\sigma_i) = \sum_j M_{ij}\sigma_j \quad \text{or} \quad M_{ij} = \frac{1}{2}\mathrm{Tr}\left[\mathcal{E}^\dagger(\sigma_i)\sigma_j\right]. \tag{39}$$

Since $\mathrm{Tr}[\mathcal{E}^\dagger(\sigma_i)\rho] = \sum_j M_{ij}\mathrm{Tr}[\sigma_j\rho]$, M_{ij} actually describes the transformation of the polarized vector $P_i \equiv \mathrm{Tr}[\sigma_j\rho]$.

Now we consider the case of two qubits under local decoherence channels, i.e., $\rho = [\mathcal{E}_A \otimes \mathcal{E}_B](\rho_0)$. To obtain the GMQD of the output state ρ through the channel, we need to get the expectation matrix $\mathcal{R}$. With the Heisenberg picture, we have

$$\mathcal{R}_{ij} = \mathrm{Tr}(\mathcal{E}_A^\dagger(\sigma_i) \otimes \mathcal{E}_B^\dagger(\sigma_j)\rho_0) = (M_A \mathcal{R}_0 M_B^T)_{ij}, \tag{40}$$

where $\mathcal{R}_0$ is the expectation matrix under ρ_0, i.e., $(\mathcal{R}_0)_{ij} = \mathrm{Tr}(\sigma_i \otimes \sigma_j \rho_0)$, and $M_{A(B)}$ is the transformation matrix characterizing the quantum channel $\mathcal{E}_{A(B)}$. So we obtain $\mathcal{R} = M_A \mathcal{R}_0 M_B^T$.

For simplicity, we assume $\mathcal{E}^A$ and $\mathcal{E}^B$ be identical, hereafter. Next, we consider three typical kinds of decoherence channels: the amplitude damping channel (ADC), the phase damping channel (PDC), and the depolarizing channel (DPC). They are described by the set of Kraus operators respectively [**?**, **?**]:

$$\begin{aligned}
K^{\mathrm{ADC}} &= \left\{\sqrt{s}|0\rangle\langle 0| + |1\rangle\langle 1|,\ \sqrt{p}|1\rangle\langle 0|\right\}, & (41)\\
K^{\mathrm{PDC}} &= \left\{\sqrt{s}\sigma_0,\ \sqrt{p}|0\rangle\langle 0|,\ \sqrt{p}|1\rangle\langle 1|\right\}, & (42)\\
K^{\mathrm{DPC}} &= \{\frac{1}{2}\sqrt{1+3s}\,\sigma_0,\ \frac{1}{2}\sqrt{p}\,\sigma_x, \frac{1}{2}\sqrt{p}\,\sigma_y,\ \frac{1}{2}\sqrt{p}\,\sigma_z\}, & (43)
\end{aligned}$$

with $s \equiv 1-p$. Here the real parameter $p \in [0,1]$ may be time-dependent in some realistic setup [**?**, **?**]. For instance, for the PDC, the parameter s may be like $\exp(-\gamma t)$ with γ the rate of damping.

From Eqs. (39), (41), (42), and (43), the transmission matrix M of each channel can be got through the transformation of the Pauli matrices in the Heisenberg picture [**?**] as

$$M_{\mathrm{ADC}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{s} & 0 & 0 \\ 0 & 0 & \sqrt{s} & 0 \\ -p & 0 & 0 & s \end{bmatrix}, \quad M_{\mathrm{PDC}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_{\mathrm{DPC}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix}. \tag{44}$$

For the density matrix under consideration, the eigenvalues of the matrix k are given by

$$\lambda_1 \equiv \lambda_1(t) = 4(|\rho_{12}(t)| + |\rho_{03}(t)|)^2$$

$$\lambda_2 \equiv \lambda_2(t) = 4(|\rho_{12}(t)| - |\rho_{03}(t)|)^2$$

$$\lambda_3 \equiv \lambda_3(t) = 2\left[(\rho_{00}(t) - \rho_{22}(t))^2 + (\rho_{11}(t) - \rho_{33}(t))^2\right]$$

finally, we find:

$$\lambda_1(t) = 4(\left|e^{-\gamma(t)}e^{-i(v_1-v_2)t}\rho_{12}\right| + \left|e^{-\gamma(t)}e^{-i(v_1+v_2)t}\rho_{03}\right|)^2 = e^{-2\gamma(t)}\lambda_1(0)$$

$$\lambda_2(t) = 4(\left|e^{-\gamma(t)}e^{-i(v_1-v_2)t}\rho_{12}\right| + \left|e^{-\gamma(t)}e^{-i(v_1+v_2)t}\rho_{03}\right|)^2 = e^{-2\gamma(t)}\lambda_2(0)$$

$$\lambda_3(t) = 2[(\rho_{00} - \rho_{22})^2 + (\rho_{11} - \rho_{33})^2] = \lambda_3(0)$$

Clearly, the eigenvalue λ_1 is larger than λ_2.

3.1 GMQD for dicke and their superpositions

we consider a state of two qubits which have been extracted from the whole ensemble. Specifically, we concentrate on the states with exchange symmetry and parity, whose two-qubit reduced density matrix can be written as

$$\rho^{AB} = \begin{pmatrix} \upsilon_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & \upsilon_- \end{pmatrix} \tag{45}$$

in the basis of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, with υ_+, υ_-. and y real, and u^*. the complex conjugate of u. The elements of the density matrix can be represented by the expectation values of the spin components

$$\upsilon_\pm = \frac{N^2 - 2N + 4\left\langle J_z^2\right\rangle \pm 4\left\langle J_z\right\rangle (N-1)}{4N(N-1)}, \tag{46}$$

$$y = \frac{N^2 - 4\left\langle J_z^2\right\rangle}{4N(N-1)}, u = \frac{\left\langle J_+^2\right\rangle}{N(N-1)} \tag{47}$$

3.1.1 Dicke states

The so-called Dicke states under consideration are defined as

$$|n\rangle_N \equiv |N/2, -N/2+n\rangle, \qquad n = 0, \ldots.., N, \tag{48}$$

As we have discussed in the previous section, to obtain the GMQD, we need to calculate the expectation values of the spin components for the stats $|n\rangle_N$, which are

$$\langle J_z\rangle = n - \frac{N}{2}, \quad \left\langle J_z^2\right\rangle = (n - \frac{N}{2})^2, \qquad \left\langle J_+^2\right\rangle = \left\langle J_-^2\right\rangle = 0. \tag{49}$$

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_3 \equiv \lambda_3(t) = \lambda_3(0) = \frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2 - N]^2}{N^2(N-1)^2}$$

$\lambda_1(t) = \lambda_2(t)$ and the maximum of the eigenvalues is $\lambda_1(t)$ or $\lambda_3(t)$

3.1.2 Generalized GHZ states

Now we discuss the GMQD of the generalized GHZ states which are superpositions of two special Dicke states. The generalized GHZ states which are important in the research of quantum mechanics are states in the following form:

$$|\psi\rangle_{GHZ} = cos\theta\,|0\rangle_N + e^{i\phi} sin\theta\,|N\rangle_N\,. \tag{50}$$

For such states, the spin expectation values are

$$\begin{aligned}\langle J_z\rangle &= -\frac{N}{2}cos2\theta,\\ \langle J_z^2\rangle &= \frac{N^2}{4},\\ \langle J_+^2\rangle &= 0.\end{aligned} \tag{51}$$

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = 0$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = 0$$

$$\lambda_3 \equiv \lambda_3(t) = 2\big[(\rho_{00}(t) - \rho_{22}(t))^2 + (\rho_{11}(t) - \rho_{33}(t))^2\big] = \lambda_3(0) = 1 + cos^2 2\theta$$

which lead to

$$D_g(\rho) = 0 \tag{52}$$

3.1.3 Superpositions of Dicke states

In this subsection, weinvestigate a more general superposition of Dicke states, which reads

$$|\psi\rangle_{SD} = cos\theta\,|n\rangle_N + e^{i\phi} sin\theta\,|n+2\rangle_N\,, \qquad n = 0, ..., N-2 \tag{53}$$

with the angle $\theta\epsilon[0, \pi)$ and relative phase $\varphi\epsilon[0, 2\pi)$. Then the expressions of the relevant spin expectation values are

$$\begin{aligned}\langle J_z\rangle &= (n - \frac{N}{2})cos^2\theta(n + 2 - \frac{N}{2})sin^2\theta,\\ \langle J_z^2\rangle &= (n - \frac{N}{2})^2 cos^2\theta(n + 2 - \frac{N}{2})^2 sin^2\theta,\\ \langle J_+^2\rangle &= \frac{1}{2}e^{i\phi} sin2\theta\sqrt{\mu_n}.\end{aligned} \tag{54}$$

Where

$$\mu_N = (n + 1)(n + 2)(N - n)(N - n - 1). \tag{55}$$

The GMQD for these states can be obtained by repeating the previous processing step. AsDG is not a function of ?, we discuss the GMQD versus ? in the following. Specifically, when n = 0

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = 4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta + \frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = 4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta - \frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_3 \equiv \lambda_3(t) = 2\big[(\rho_{00}(t) - \rho_{22}(t))^2 + (\rho_{11}(t) - \rho_{33}(t))^2\big] = (1 - \frac{4}{N}sin^2\theta)^2 + [1 - \frac{8(N-2)}{N(N-1)}sin^2\theta]^2$$

4 numerical analysis

4.1 Dicke states

For the density matrix under consideration, the eigenvalues of the matrix k are given by

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}$$

$$\lambda_3 \equiv \lambda_3(t) = \lambda_3(0) = \frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2 - N]^2}{N^2(N-1)^2}$$

$\lambda_1(t) = \lambda_2(t)$ and the maximum of the eigenvalues is $\lambda_1(t)$ or $\lambda_3(t)$

Therefore, for these X-states, equation (23) is reduced to a simpler form

$$D_G(\rho) = \frac{1}{4}(min\{\lambda_1(t), \lambda_3(t)\} + \lambda_2(t)) \tag{56}$$

Specifically, When $N = 15 \quad n = 0, ..., 15$

- For $n = 0, ..., 4$ and $n = 11, ..., 15$

  The geometric quantum discord ρ_{12} is

$$D_\rho = \frac{1}{2}(\lambda_1) = \frac{1}{2}e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2} \tag{57}$$

  when $\lambda_3 > \lambda_1$ (*)

- For $n = 5$ and $n = 10$ the condition (*) is satisfied for $t_0 < t$. In this situation, the geometric quantum discord is

$$D_G = \frac{1}{2}e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2} \tag{58}$$

  when the transmission parameter t satisfies $t_0 > t$

  we have $\lambda_3 < \lambda_1$ and the geometric quantum discord is given by

$$D_G = \frac{1}{4}(\lambda_3 + \lambda_2) = \frac{1}{4}(\frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2 - N]^2}{N^2(N-1)^2} + e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}) \tag{59}$$

- For $n = 6, 7$ and $n = 8, 9$

  the geometric quantum discord ρ_{12} is

$$D_G = \frac{1}{4}(\lambda_3 + \lambda_2) = \frac{1}{4}(\frac{(N-2n)^2}{N^2} + \frac{[(N-2n)^2 - N]^2}{N^2(N-1)^2} + e^{-2\gamma(t)}\frac{4n^2(N-n)^2}{N^2(N-1)^2}) \tag{60}$$

  when$\lambda_3 < \lambda_1$

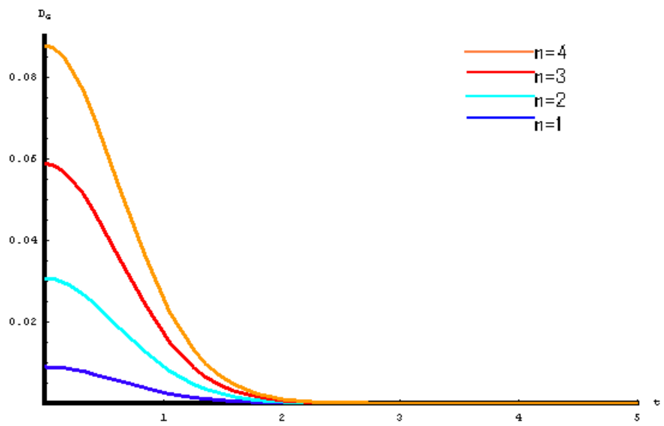
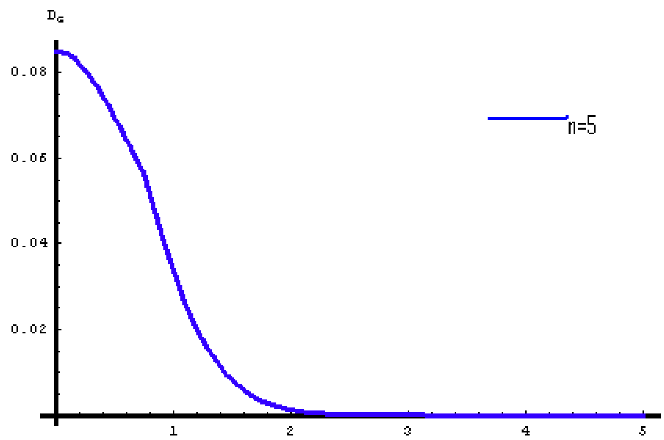
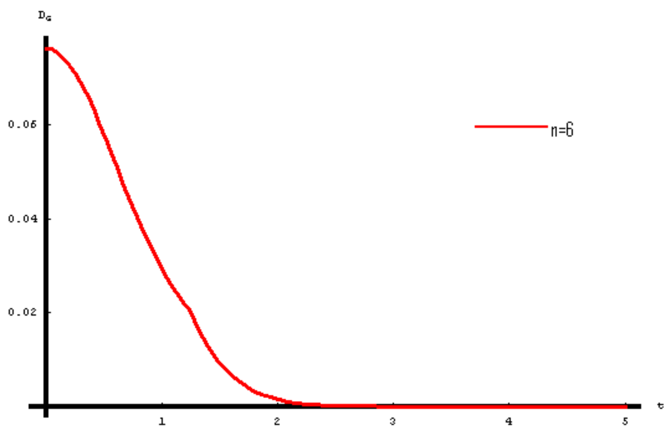


Figure 1: (a) Dynamics of GMQD for the sub-Ohmic reservoirs of Dicke states with N fixed $(N = 15)$ and $(n = 1, .., 4)$ $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).





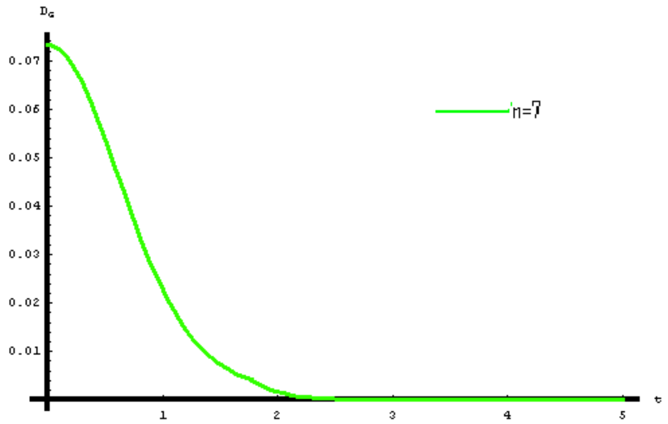


Figure 1: (b) Dynamics of GMQD for the sub-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 5, 6, 7$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).

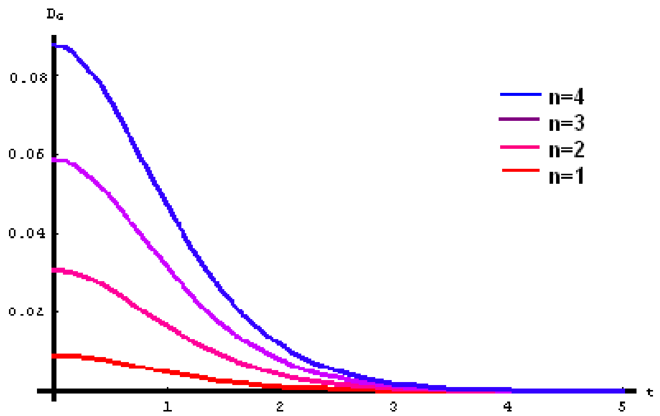
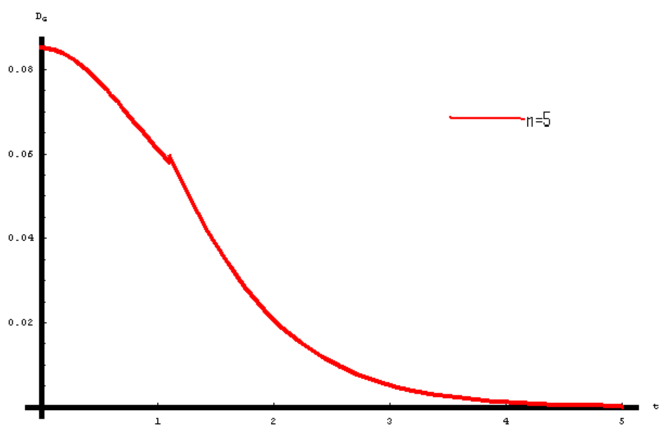


Figure 2: (a) Dynamics of GMQD for the Ohmic reservoirs of dicke states with N fixed ($N = 15$) and ($n = 1, .., 4$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).



.

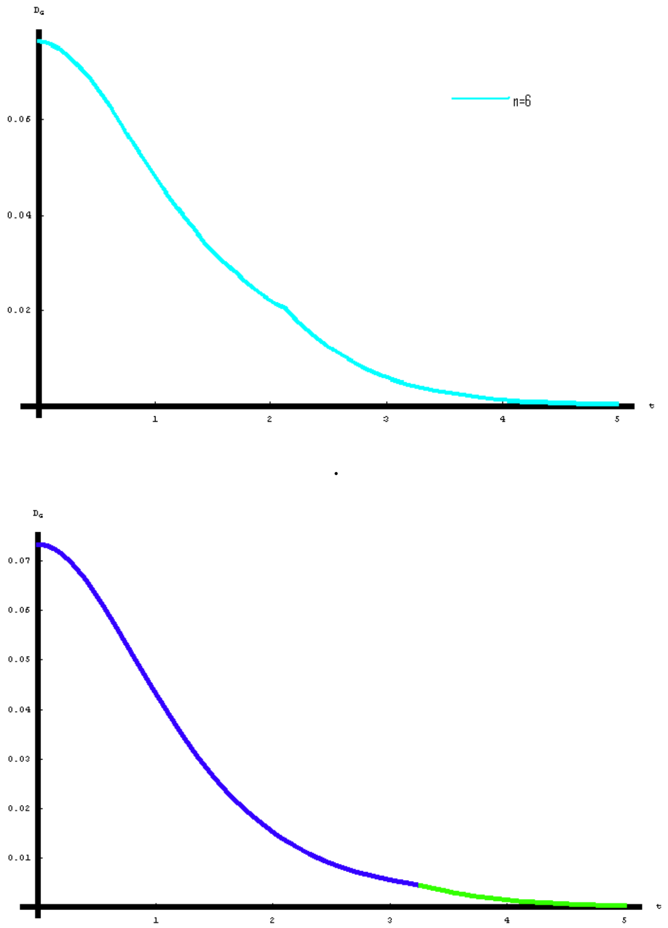


Figure 2: (b) Dynamics of GMQD for the Ohmic reservoirs of dicke states with N fixed ($N = 15$) and ($n = 5, 6, 7$) $s = 0.5, \lambda = 0.1, \Omega\beta = 1$ (numerical calculation with the upper bound of m being 10^5).

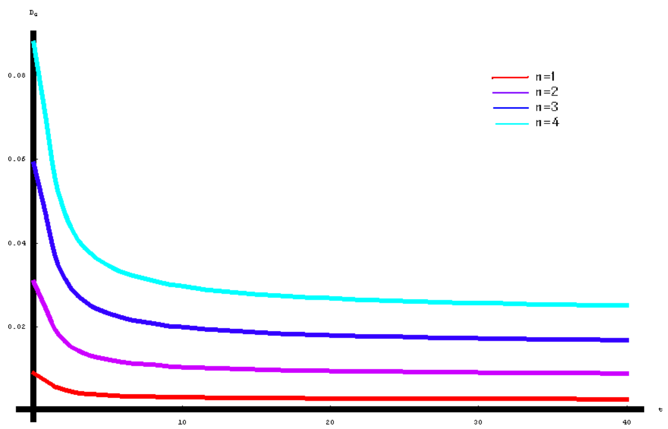


Figure 3: (a) Dynamics of GMQD for the super-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 1, .., 4$) $s = 1.5, \lambda = 0.2$ (numerical calculation with the upper bound of m being 10^5).

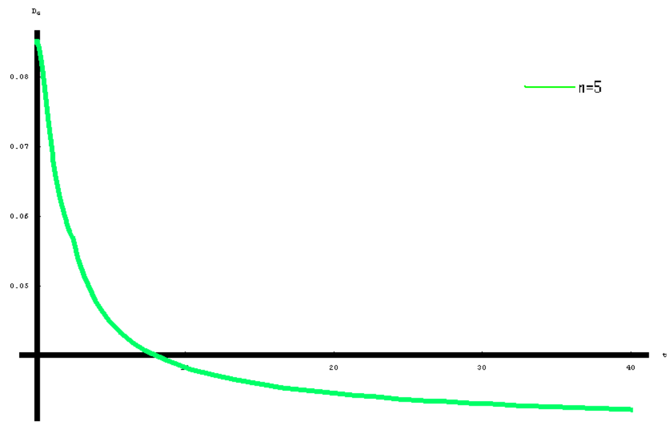


Figure 3: (b) Dynamics of GMQD for the super-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 5$) $s = 1.5, \lambda = 0.2$ (numerical calculation with the upper bound of m being 10^5).

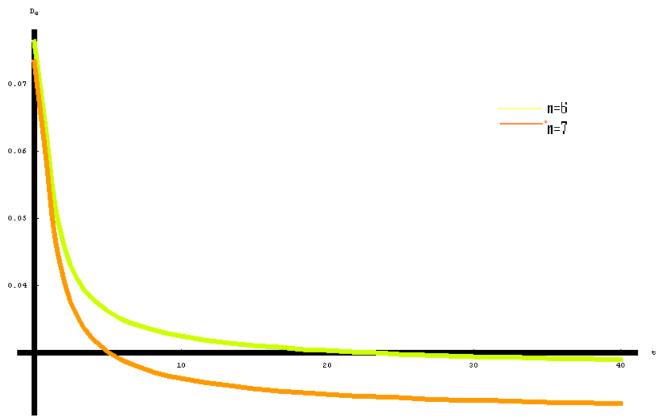


Figure 3: (c)Dynamics of GMQD for the super-Ohmic reservoirs of Dicke states with N fixed ($N = 15$) and ($n = 0, .., 15$) $s = 1.5, \lambda = 0.2$ (numerical calculation with the upper bound of m being 10^5).

It can also be seen that there are two classes of the evolution of GMQD. (a)D_G is a monotonic decreasing function of time t with the limit zero, if $\lambda_3 > \lambda_1$. (b)D_G is a piecewise monotonic decreasing function with one turning point at the time $t = \bar{t}$ defined by $\lambda_3 = \lambda_1$ and then has the limit zero as t tends to infinity, if $\lambda_3 < \lambda_1$. These two classes of the evolution of D_G are plotted in fig 1 and 2.

In the figure 3, there are two possible classes of the evolution of D_G.(a)and (b) D_G is a monotonic decreasing function of time with a constant limit $(1/2)\lambda_1$ or $(1/4)(\lambda_3 + \lambda_2)$, if $\lambda_3 > \lambda_1$, or if $\lambda_3 < \lambda_1$. (b)D_G is a piecewise monotonic decreasing function with one turning point at the time $t = \bar{t}$ defined by $\lambda_3 = \lambda_1$, and then has a constant limit as t tends to infinity, if $\lambda_3 < \lambda_1$.

4.2 Superpositions of Dicke states

For the density matrix under consideration

$$\lambda_1 \equiv \lambda_1(t) = e^{-2\gamma(t)}\lambda_1(0) = 4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta + \frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_2 \equiv \lambda_2(t) = e^{-2\gamma(t)}\lambda_2(0) = 4e^{-2\gamma(t)}[\frac{2(N-2)}{N(N-1)}sin^2\theta - \frac{|sin2\theta|}{\sqrt{2N(N-1)}}]^2$$

$$\lambda_3 \equiv \lambda_3(t) = 2\big[(\rho_{00}(t)-\rho_{22}(t))^2 + (\rho_{11}(t)-\rho_{33}(t))^2\big] = (1-\frac{4}{N}sin^2\theta)^2 + [1-\frac{8(N-2)}{N(N-1)}sin^2\theta]^2 \quad (61)$$

It is noted that D_G is a periodic function of θ with period π. Accordingly, we plot the GMQD versus ? within one period. To compare the eigenvalues λ_1 and λ_3, we find that there are two cases.

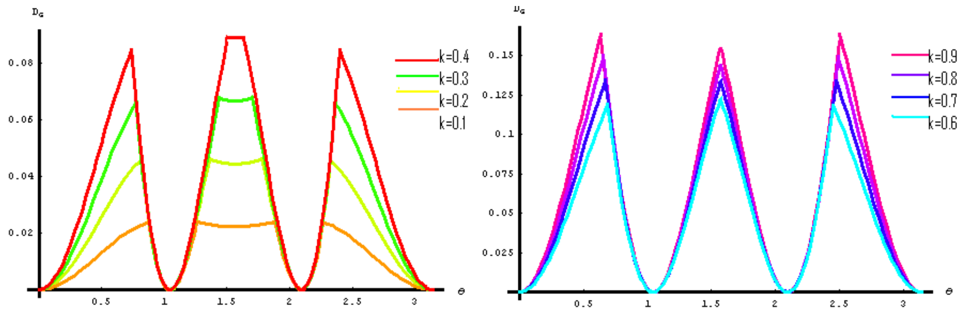


Figure 5: The GMQD of the superpositions of Dicke states verus θ when $n = 0$ with $N = 3$ and $k = 0.1, ...0.9$.

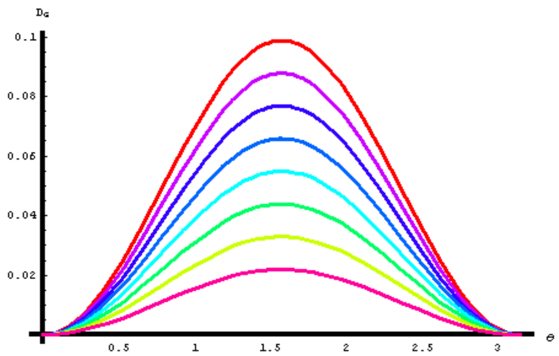


Figure 7: The GMQD of the superpositions of Dicke states verus θ when $n = 0$ with $N = 8$ and $k = 0.1, ...0.9$.

When $N < 8$, as can be seen in figure 4 (N=3), the maximum eigenvalue is λ_1 in the middle region and λ_3 at the edge near 0 and π for $k = 0.6, 0.7, 0.8, 0.9$ and for $k = 0.1, 0.2, 0.3, 0.4$ the maximum eigenvalue is λ_3 in the middle region and at the edge near 0 and π and λ_1 between the middle region and at the edge near 0.

In figure 5, there are three local maxima of D_G. The first and third maxima display just at the point when $\lambda_2 = \lambda_3$. While the second one accurs at $\theta = \frac{\pi}{2}$ which is due to the symmetry of equation (61). We also observe that there exist two special values of θ, $\theta = \pi/3$ and $\theta = 2\pi/3$, where the GMQD is zero. These zero values of the GMQD are induced by $\lambda_2 = \lambda_3 = 0$ at these two points (while λ_1 is the maximum).

However,When $N \geq 8$ as is shown in fig 6, $\lambda_3 > \lambda_1$ no matter what θ is. Consequently, D_G has only one maximum at $\theta = \pi/2$, as is shown in figure 7, and it can analytically be written as

$$D_G = e^{-2\gamma(t)}[\frac{8(N-2)^2 sin^4\theta}{N^2(N-1)^2} + \frac{sin^2 2\theta}{N(N-1)}] \tag{62}$$

5 Conclusion

In this paper, in order to investigate the pairwise quantum correlations, we have studied the dynamics of GMQD The model under consideration consists of two qubits coupled with two independent bosonic reservoirs described by Ohmic-like spectral densities. The Hamiltonian of the model is described as Eq (7), and the two-qubit system is initially in the X states decoupled from the environments, described by Eq (8). We examine the evolution of GMQD for the three types of reservoirs, sub-ohmic, ohmic, and super-ohmic particularly the Dicke states and their superpositions. D_G is a monotonic decreasing function of time t or a poiecewise monotonic decreasing function with one turning point before becoming frozen phenomenon.

References

[1] C. H. Bennett, Phys. Rev. Lett. 68, 3121 (1992).

[2] C. H. Bennett et al., Phys. Rev. Lett. 70, 1895 (1993).

[3] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).

[4] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[5] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, (2000), Chap. 3.

[6] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).

[7] E. Knill and R. Laflamme, Phys. Rev. Lett. 81, 5672 (1998).

[8] Groisman B, Popescu S and Winter A 2005 Phys. Rev. A 72 032317.

[9] Dakic B, Vedral V and Brukner C 2010 Phys. Rev. Lett. 105190502.

[10] Daoud M and Ahl Laamara R 2012 Phys. Lett. A 376 2361.

[11] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).

[12] L. Henderson and V. Vedral, J. Phys. A 34, 6899 (2001).

[13] V. Vedral, Phys. Rev. Lett. 90, 050401 (2003).

[14] J. Maziero, L. C. Celèri, R. M. Serra, and V. Vedral, Phys, Rev A 80, 044102 (2009).

[15] S. Luo, Phys. Rev. A 77, 042303 (2008).

[16] T. Werlang, S. Souza, F. F. Fanchini, and C. J. Villas Boas, Phys. Rev. A 80, 024103 (2009).

[17] R. F. Werner, Phys. Rev. A 40, 4277 (1989).

[18] D. Girolami, and G. Adesso, Phys. Rev. A 83, 052108 (2011).