Distribution of geometric quantum discord in photon-added coherent states

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Abstract

We examine the monogamy relation of geometric quantum discord in photon added coherent states of Greenberger-Horne-Zeilinger. The Hilbert-Schmidt norm is used as quantifier of pairwise quantum correlations. The geometric quantum discord in all bipartite subsystems are explicitly given. The behavior of geometric quantum discord and its monogamy property versus the excitation photon number are discussed.

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1 Introduction

Entangled coherent states have found various applications in quantum information science (for a recent review see [1]). In fact, they were shown to serve as valuable resource in quantum teleportation [2, 3, 4, 5, 6, 7], quantum networks [8] , quantum logical encoding [9], quantum computation [10], quantum information processing [11], and quantum metrology [12, 13]. The experimental production of superposed coherent states constitutes in general a challenging task. However, despite their extreme sensitivity to environmental effects, several schemes were proposed in the literature for their generation [14, 15, 16, 17, 18]. One may quote for example the generation of entangled coherent states in atomic Bose-Einstein condensates [19]. Usually coherent states are treated as continuous variable states. Recently, the idea of encoding quantum information on coherent states has led to an interesting proposal [?] in which Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ ($\alpha \in \mathbb{C}$) are used to encode logical qubits. In this scenario, balanced superpositions of n-partite Glauber coherent states of type

$$|\alpha, \alpha, \cdots, \alpha\rangle \pm |-\alpha, -\alpha, , \cdots, -\alpha\rangle$$

can be mapped in n qubit states. They reduce for $n = 2$ to bipartite states commonly termed in the literature quasi-Bell coherent states in analogy with the four Bell states defined for two dimensional quantum systems. Also, in the special $n = 3$, one recovers the quasi-GHZ coherent states which are the non orthogonal extensions of the usual Greenberger-Horne-Zeilinger three qubit states. Entanglement properties of quasi-Bell and quasi-GHZ coherent states were initially investigated using the formalism of concurrence or equivalently entanglement of formation. Quantum discord which goes beyond entanglement of formation was also considered to evaluate the pairwise quantum discord in multipartite coherent states. It is also important to emphasize that the geometric variant of quantum discord, based on the notion of Hilbert-Schmidt norm, was used as quantifier of bipartite correlations in such states.

In the same spirit, using the formalism of photon added coherent states, bipartite correlations in single mode excited entangled quasi-Bell and quasi-GHZ coherent states were investigated in ????. The pairwise quantum correlations are quantified by means of Wootters concurrence. In this paper, we shall be concerned with the derivation of pairwise geometric quantum discord in photon added quasi-GHZ coherent states in order to investigate the distribution of quantum correlation among the three modes. At this stage, it worth to notice that in multipartite quantum systems, one of the most important properties is the monogamy relation which limits the free shareability and subsequently imposes severe restrictions on the distribution of quantum correlations between the different parts of the system. This concept was first discussed by Coffman, Kundo and Wootters in 2001 [?] for the entanglement of formation in three qubits and latter generalized for N qubits [?]. It was also extended to other measures (quantum discord. geometric quantum discord, ...) of quantum correlations [?],[?], [?],[?], [?],[?].

The paper is organized as follows. In section 2, we introduce photon added coherent states of Greenberger-Horne-Zeilinger type which interpolate continuously between the . In particular we discuss the different bi-partitions of such tripartite states. Two scenario are considered: pure and mixed. In each case, a suitable qubit mapping is defined. The pairwise geometric discord is derived in section 3. The influence of the photon addition process is discussed. A special attention is devoted to the limiting case corresponding to photon added states of W-type. The monogamy relation of geometric discord is examined in section 3 and the effects of the excitation are discussed. Concluding remarks close this paper.

2 Adding photons to quasi-GHZ coherent states and three qubit encoding

2.1 Excitations of quasi-GHZ coherent states

We are interested in tripartite quantum states involving Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ with the same amplitude α and phases differing by π. In this paper, we consider the GHZ-type entangled coherent states of the form

$$|\mathrm{GHZ}_k(\alpha)\rangle = \mathcal{C}_k(\alpha)(|\alpha,\alpha,\alpha\rangle + e^{ik\pi}|-\alpha,-\alpha,-\alpha\rangle). \tag{1}$$

where the normalization constant $\mathcal{C}_k$ is given by

$$\mathcal{C}_k^{-2}(\alpha) = 2 + 2e^{-6|\alpha|^2}\cos k\pi. \tag{2}$$

A single-mode of electromagnetic field is algebraically described by the Weyl-Heisenberg algebra spanned by creation a^+ and annihilation a^- operators. The adding photons process is mathematically realized through successive applications of a^+. Thus, m successive actions of creation operator a^+ on the Glauber coherent states $|\alpha\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{3}$$

leads to the un-normalized states

$$||\alpha,m\rangle = \left(a^+\right)^m \mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\sqrt{(n+m)!}\mid n+m\rangle. \tag{4}$$

The vectors $|n\rangle$ denote the Fock-Hilbert states of harmonic oscillator. The normalized m-photon added coherent states are defined by

$$|\alpha,m\rangle = \frac{(a^+)^m|\alpha\rangle}{\sqrt{\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle}}, \tag{5}$$

where

$$\langle\alpha|(a^-)^m(a^+)^m|\alpha\rangle = m!L_m(-|\alpha|^2). \tag{6}$$

In the last expression $L_m(x)$ is the Laguerre polynomial of order m defined by

$$L_m(x) = \sum_{n=0}^{m}\frac{(-1)^n m! x^n}{(n!)^2(m-n)!}. \tag{7}$$

Photon added coherent states interpolate between electromagnetic field coherent states (quasi-classical states) and Fock states $|n\rangle$ (purely quantum states). Furthermore, photon added coherent states exhibit non-classical features such as squeezing, negativity of Wigner distribution and sub Poissonian statistics. Their experimental generation using parametric down conversion in a nonlinear crystal was reported in [**?**]. Photon-coherent states are not orthogonal each other. Indeed using the expression

$$\langle -\alpha|(a^-)^m(a^+)^m|\alpha\rangle = e^{-2|\alpha|^2} m! L_m(|\alpha|^2), \tag{8}$$

it is simply verified that the overlapping between the states $|\alpha, m)$ and $|-\alpha, m)$ is

$$\langle -\alpha, m|\alpha, m\rangle = e^{-2|\alpha|^2} \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{9}$$

The excitation of the first mode of the state (1), by adding m photon, leads to the tripartite state

$$||\mathrm{GHZ}_k(\alpha, m)\rangle = ((a^+)^m \otimes \mathbb{I} \otimes \mathbb{I})\ |\mathrm{GHZ}_k(\alpha)\rangle, \tag{10}$$

from which we introduce the normalized photon added quasi-GHZ coherent states as

$$|\mathrm{GHZ}_k(\alpha, m)\rangle = \frac{||\mathrm{GHZ}_k(\alpha, m)\rangle}{\sqrt{\langle \mathrm{GHZ}_k(\alpha, m)||\mathrm{GHZ}_k(\alpha, m)\rangle}}. \tag{11}$$

Using the expressions (6) and (8), the vector (11) rewrites as

$$|\mathrm{GHZ}_k(\alpha, m)\rangle = \mathcal{C}_k(\alpha, m)(|m, \alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}|m, -\alpha\rangle \otimes |-\alpha\rangle \otimes |-\alpha\rangle). \tag{12}$$

where the normalization factor is

$$\mathcal{C}_k^{-2}(\alpha, m) = 2 + 2\kappa_m e^{-6|\alpha|^2} \cos k\pi. \tag{13}$$

with

$$\kappa_m \equiv \kappa_m(|\alpha|^2) := \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{14}$$

The quantity κ_m, defined by (14), goes to unit for $m = 0$ so that the state $|\mathrm{GHZ}_k(\alpha, m)\rangle$ (12) reduces to $|\mathrm{GHZ}_k(\alpha)\rangle$ (1). It is also important to note that for $|\alpha|$ large, the overlap between Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ approaches zero and then they are quasi-orthogonal. In this limiting situation the state $|\mathrm{GHZ}_k(\alpha)\rangle$ (1) reduces to an usual three qubit state of GHZ-type

$$|\mathrm{GHZ}_k(\infty)\rangle = \frac{1}{\sqrt{3}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + e^{ik\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle). \tag{15}$$

where $|\mathbf{0}\rangle \equiv |\alpha\rangle$ and $|\mathbf{1}\rangle \equiv |\alpha\rangle$.

2.2 Qubit encoding

In investigating the pairwise quantum discord in a tripartite system $1-2-3$ described by a quantum state of type $|\mathrm{GHZ}_k(\alpha,m)\rangle$, one needs the reduced density matrices describing the two qubit subsystems $1-2$, $2-3$ and $1-3$. For the states $|\mathrm{GHZ}_k(\alpha,m)\rangle$ (12), it is simply seen that the reduced density matrices $\rho_{12} = \mathrm{Tr}_3\rho_{123}$ and $\rho_{13} = \mathrm{Tr}_2\rho_{123}$ are identical. The pure three mode density matrix ρ_{123} is given

$$\rho_{123} = |\mathrm{GHZ}_k(\alpha,m)\rangle\langle \mathrm{GHZ}_k(\alpha,m)|.$$

The reduced density matrices ρ_{12} and ρ_{13} are given by

$$\rho_{12} = \rho_{13} = \frac{\mathcal{C}_k^2(\alpha,m)}{\mathcal{N}_k^2(\alpha,m)}\left[\left(\frac{1+e^{-2|\alpha|^2}}{2}\right)|\mathrm{B}_k(\alpha,m)\rangle\langle \mathrm{B}_k(\alpha,m)| + \left(\frac{1-e^{-2|\alpha|^2}}{2}\right)Z|\mathrm{B}_k(\alpha,m)\rangle\langle \mathrm{B}_k(\alpha,m)|Z\right] \quad (16)$$

in terms of photon added quasi-Bell states defined by

$$|\mathrm{B}_k(\alpha,m)\rangle = \mathcal{N}_k(\alpha,m)\ \big[|m,\alpha\rangle\otimes|\alpha\rangle + e^{ik\pi}\ |m,-\alpha\rangle\otimes|-\alpha\rangle\big], \quad (17)$$

in terms of the normalized photon added coherent state (5), with

$$\mathcal{N}_k^{-2}(\alpha,m) = 2 + 2\kappa_m e^{-4|\alpha|^2}\cos k\pi. \quad (18)$$

where κ_m is defined by (14). For $m=0$, the quasi-Bell states

$$|\mathrm{B}_k(\alpha)\rangle = \mathcal{N}_k(\alpha,0)\ \big[|\alpha\rangle\otimes|\alpha\rangle + e^{ik\pi}|-\alpha\rangle\otimes|-\alpha\rangle\big] \quad (19)$$

are recovered. The operator Z, in (16), is the third Pauli generator defined by

$$Z|\mathrm{B}_k(\alpha,m)\rangle = \mathcal{N}_k(\alpha,m)[|m,\alpha\rangle\otimes|\alpha\rangle - e^{ik\pi}|m,-\alpha\rangle\otimes|-\alpha\rangle]$$

Similarly, by tracing out the third mode, the reduced matrix density ρ_{23} takes the form

$$\rho_{23} = \frac{\mathcal{C}_k^2(\alpha,m)}{\mathcal{N}_k^2(\alpha,0)}\left[\left(\frac{1+\kappa_m e^{-2|\alpha|^2}}{2}\right)|\mathrm{B}_k(\alpha,0)\rangle\langle \mathrm{B}_k(\alpha,0)| + \left(\frac{1-\kappa_m e^{-2|\alpha|^2}}{2}\right)Z|\mathrm{B}_k(\alpha,0)\rangle\langle \mathrm{B}_k(\alpha,0)|Z\right] \quad (20)$$

To derive the pairwise correlation between the components of the subsystems $1-2$, $2-3$ and $1-3$, we assume that the information is encoded in even and odd Glauber coherent states (Shrödinger cat states). In this sense, we introduce for the first mode the following qubit mapping

$$|m,\pm\alpha\rangle = \sqrt{\frac{1+\kappa_m e^{-2|\alpha|^2}}{2}}\ |0\rangle_1 \pm \sqrt{\frac{1-\kappa_m e^{-2|\alpha|^2}}{2}}\ |1\rangle_1 \quad (21)$$

for the first mode. For the second and third modes, we consider the qubits defined by

$$|\pm\alpha\rangle = \sqrt{\frac{1+e^{-2|\alpha|^2}}{2}}\ |0\rangle_i \pm \sqrt{\frac{1-e^{-2|\alpha|^2}}{2}}\ |1\rangle_i \qquad i=2,3 \quad (22)$$

Substituting (21) and (22) in (16) (resp. (20)), one can express the density matrix ρ_{12} (resp. ρ_{23}) in the two qubit basis $\{|0\rangle_1\otimes|0\rangle_2, |0\rangle_1\otimes|1\rangle_2, |1\rangle_1\otimes|0\rangle_2, |1\rangle_1\otimes|1\rangle_2\}$ (resp. $\{|0\rangle_2\otimes|0\rangle_3, |0\rangle_2\otimes|1\rangle_3, |1\rangle_2\otimes$

$|0\rangle_3, |1\rangle_2 \otimes |1\rangle_3\}$). The resulting density matrices, in this encoding scheme, have non vanishing entries only along the diagonal and the anti-diagonal.

Usually coherent states are treated as continuous variable states. Recently, the idea of encoding quantum information on coherent states has led to an interesting proposal [**?**] in which superpositions of Glauber coherent states are used to encode logical qubits. Accordingly, one can consider an encoding scheme of type $|\alpha\rangle \longrightarrow |0\rangle$ and $|-\alpha\rangle \longrightarrow |1\rangle$ with two non orthogonal logical qubits. Alternatively, an orthogonal qubit encoding scheme involving even and odd Glauber coherent states can be be defined so that the coherent states are mapped in in $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space. Hence, for photon added coherent of type (5), one introduces the two dimensional basis spanned by two orthogonal qubits $|+, m\rangle$ and $|-, m\rangle$ defined as even and odd superpositions of photon added coherent states (5)

$$|\pm, m\rangle = \frac{1}{\sqrt{2 \pm 2\kappa_m e^{-2|\alpha|^2}}} \left(|\alpha, m) \pm |-\alpha, m)\right) \tag{23}$$

Clearly, for $m = 0$, one has $\kappa_0 = 1$ and the logical qubits (23) reduce to

$$|\pm\rangle = \frac{1}{\sqrt{2 \pm 2e^{-2|\alpha|^2}}} \left(|\alpha\rangle \pm |-\alpha\rangle\right), \tag{24}$$

which coincide with even and odd Glauber coherent states providing the qubit encoding scheme introduced in [**?**]. It must be emphasized that such qubit encoding of paramount importance in dealing with quantum correlation in photon added coherent states and to investigate the influence of the photon adding excitation process. In this, we shall first consider the entanglement in quasi-Bell states. The quasi-Bell states are very interesting in quantum optics and serve as valuable resource for quantum teleportation and many others quantum computing operations.

bipartition pure

we introduce the orthogonal basis $\{|0\rangle_1, |1\rangle_1\}$ defined by

$$|0\rangle_1 = \frac{|\alpha, m\rangle + |\alpha, m\rangle}{\sqrt{2(1 + \kappa_m e^{-2|\alpha|^2})}} \qquad |1\rangle_1 = \frac{|\alpha, m\rangle - |\alpha, m\rangle}{\sqrt{2(1 - \kappa_m e^{-|\alpha|^2})}}, \tag{25}$$

for the first subsystem. For the modes (23), grouped into a single subsystem, we introduce the orthogonal basis $\{|0\rangle_{23}, |1\rangle_{23}\}$ given by

$$|0\rangle_{23} = \frac{|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle}{\sqrt{2(1 + e^{-4|\alpha|^2})}} \qquad |1\rangle_{23} = \frac{|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle}{\sqrt{2(1 + e^{-4|\alpha|^2})}}. \tag{26}$$

Inserting(25) and (26) in $\mathrm{GHZ}_k(\alpha, m)$, we get the form of the pure state $\mathrm{GHZ}_k(\alpha, m)$ in the basis $\{|0\rangle_1 \otimes |0\rangle_{23}, |0\rangle_1 \otimes |1\rangle_{23}, |1\rangle_1 \otimes |0\rangle_{23}, |1\rangle_1 \otimes |1\rangle_{23}\}$. Explicitly, it is given by

$$|\mathrm{GHZ}_k(\alpha, m)\rangle = \sum_{\alpha=0,1} \sum_{\beta=0,1} C_{\alpha,\beta} |\alpha\rangle_1 \otimes |\beta\rangle_{23} \tag{27}$$

where the coefficients $C_{\alpha,\beta}$ are

$$C_{0,0} = \mathcal{C}_k(\alpha, m)(1 + e^{ik\pi})c_1^+ c_{23}^+, \qquad C_{0,1} = \mathcal{C}_k(\alpha, m)(1 - e^{ik\pi})c_1^+ c_{23}^-$$

$$C_{1,0} = \mathcal{C}_k(\alpha, m)(1 - e^{ik\pi})c_{23}^+ c_1^-, \qquad C_{1,1} = \mathcal{C}_k(\alpha, m)(1 + e^{ik\pi})c_1^- c_{23}^-.$$

in terms of the quantities

$$c_1^{\pm} = \sqrt{\frac{1 \pm \kappa_m e^{-2|\alpha|^2}}{2}} \qquad c_{23}^{\pm} = \sqrt{\frac{1 \pm e^{-4|\alpha|^2}}{2}}$$

3 Geometric measure of quantum discord in photon added coherent states

3.1 Closest classical states to two qubit X states

To begin, we shall present the procedure leading to the closest classically correlated state to the two-qubit X state given by

$$\rho_{12} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{28}$$

The state (28) reads in Fano-Bloch representation as

$$\rho_{12} = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + T_{03}\sigma_0 \otimes \sigma_3 + T_{30}\sigma_3 \otimes \sigma_0 + \sum_{kl} T_{kl}\sigma_k \otimes \sigma_l\right] \tag{29}$$

where the correlation matrix elements are given by ??????. The geometric measure of quantum discord is defined as the distance the state ρ_{12} and its closest classical-quantum state presenting zero discord [?]

$$D_{\rm g}(\rho_{12}) = \min_{\chi_{12}} ||\rho_{12} - \chi_{12}||^2 \tag{30}$$

where the Hilbert-Schmidt norm is defined by $||X||^2 = {\rm Tr}(X^\dagger X)$ and the minimization is taken over the set of all classical states. When the measurement is performed on the qubit 1, the classical states write

$$\chi_{12} = p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1^2 + p_2|\psi_2\rangle\langle\psi_2| \otimes \rho_2^2 \tag{31}$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis related to the qubit 1, p_i $(i = 1, 2)$ stands for probability distribution and ρ_i^2 $(i = 1, 2)$ is the marginal density of the qubit 2. The classically correlated states χ_{12} can also be written as

$$\chi_{12} = \frac{1}{4}\left[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} te_i \;\; \sigma_i \otimes \sigma_0 + \sum_{i=1}^{3}(s_+)_i \;\sigma_0 \otimes \sigma_i + \sum_{i,j=1}^{3} e_i(s_-)_j \;\sigma_i \otimes \sigma_j\right] \tag{32}$$

where

$$t = p_1 - p_2, \qquad e_i = \langle\psi_1|\sigma_i|\psi_1\rangle, \qquad (s_\pm)_j = {\rm Tr}\big((p_1\rho_1^2 \pm p_2\rho_2^2)\sigma_j\big).$$

It follows that the distance between the density matrix ρ_{12} and the classical state χ_{12}, as measured by Hilbert-Schmidt norm, is then given by

$$||\rho_{12} - \chi_{12}||^2 = \frac{1}{4}\left[(t^2 - 2te_3T_{30} + T_{30}^2) + \sum_{i=1}^{3}(T_{0i} - (s_+)_i)^2 + \sum_{i,j=1}^{3}(T_{ij} - e_i(s_-)_j)^2\right] \tag{33}$$

The minimization of the distance (33), with respect to the parameters t, $(s_+)_i$ and $(s_-)_i$, gives

$$t = e_3 T_{30}$$

$$(s_+)_1 = 0 \quad (s_+)_2 = 0 \quad (s_+)_3 = T_{03}$$

$$(s_-)_i = \sum_{j=1}^{3} e_j T_{ji}. \tag{34}$$

Inserting these solutions in (33), one has

$$||\rho_{12} - \chi_{12}||^2 = \frac{1}{4}\Big[\mathrm{Tr}K - \vec{e}^{\,t} K \vec{e}\Big] \tag{35}$$

where the matrix K is defined by

$$K = xx^\dagger + TT^\dagger \tag{36}$$

with

$$x^\dagger = (0, 0, T_{30}) \qquad T = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}.$$

From equation (35), one see that the minimal value of Hilbert-Schmidt distance (35) is reached for the largest eigenvalue of the matrix K. We denote by λ_1, λ_2 and λ_3 the eigenvalues of the matrix K (36) corresponding to the X state (28) or equivalently (29). They are given by

$$\lambda_1 = 4(|\rho_{14}| + |\rho_{23}|)^2, \qquad \lambda_2 = 4(|\rho_{14}| - |\rho_{23}|)^2, \qquad \lambda_3 = 2[(\rho_{11} - \rho_{33})^2 + (\rho_{22} - \rho_{44})^2]. \tag{37}$$

To get the minimal value of the Hilbert-Schmidt distance (35) and subsequently the amount of geometric quantum discord, one compares λ_1, λ_2 and λ_3. As λ_1 is always greater than λ_2, the largest eigenvalue $\lambda_{\max}$ is λ_1 or λ_3. It follows that the geometric discord is given by

$$D_g(\rho_{12}) = \frac{1}{4}\min\{\lambda_1 + \lambda_2, \lambda_2 + \lambda_3\}. \tag{38}$$

To write down the explicit expressions of the closest classical state χ_{12} to ρ_{12}, one has to determine the eigenvector $\vec{e}_{\max}$ associated with the largest eigenvalue $\lambda_{\max}$. In this respect, two cases ($\lambda_{\max} = \lambda_1$ and $\lambda_{\max} = \lambda_3$) are separately discussed. We begin by density matrices ρ_{12} (28) whose entries satisfy the condition $\lambda_{\max} = \lambda_3$. The associated eigenvector is given by $\vec{e}_3 = (0, 0, 1)$. Replacing in the set of constraints (34), one has

$$\chi^3_{12} = \frac{1}{4}\Big[\sigma_0 \otimes \sigma_0 + T_{30} \ \ \sigma_3 \otimes \sigma_0 + T_{03} \ \sigma_0 \otimes \sigma_3 + T_{33} \ \sigma_3 \otimes \sigma_3\Big] \tag{39}$$

In the second situation, the eigenvector corresponding to λ_1 is given by $\vec{e}_1 = (\cos\frac{\phi}{2}, -\sin\frac{\phi}{2}, 0)$ where $e^{i\phi} = \frac{\rho_{14}\rho_{23}}{|\rho_{14}||\rho_{23}|}$. Reporting the components of $\vec{e}_1$ in (34), one gets the closest classical state

$$\chi^1_{12} = \frac{1}{4}\Big[\sigma_0 \otimes \sigma_0 + T_{30} \ \ \sigma_3 \otimes \sigma_0 + \sum_{i=1}^{2}\sum_{j=1}^{2} \tilde{T}_{ij} \ \sigma_i \otimes \sigma_j\Big] \tag{40}$$

where

$$\tilde{T}_{11} = \cos\frac{\phi}{2}(\cos\frac{\phi}{2}T_{11} - \sin\frac{\phi}{2}T_{21}) \qquad \tilde{T}_{12} = \cos\frac{\phi}{2}(\cos\frac{\phi}{2}T_{12} - \sin\frac{\phi}{2}T_{22})$$

$$\tilde{T}_{21} = -\sin\frac{\phi}{2}(\cos\frac{\phi}{2}T_{11} - \sin\frac{\phi}{2}T_{21}) \qquad \tilde{T}_{22} = -\sin\frac{\phi}{2}(\cos\frac{\phi}{2}T_{12} - \sin\frac{\phi}{2}T_{22}).$$

3.2 Pairwise geometric discord in tripartite quasi-GHZ coherent states

In this section, we are interested to study the tripartite quantum correlations present in single mode excited entangled coherent states (12). Using the qubit mapping (23) and (24), the bipartite density matrix $\rho_{12} = \mathrm{Tr}_3\rho_{123}$ writes in Fano-Bloch form as

$$\rho_{12} = \frac{1}{4}\left[\sigma_0\otimes\sigma_0 + R^{12}_{03}\sigma_0\otimes\sigma_3 + R^{12}_{30}\sigma_3\otimes\sigma_0 + \sum_{k=1,2,3} R^{12}_{kk}\sigma_k\otimes\sigma_k\right] \tag{41}$$

$$\begin{aligned}
R^{12}_{00} &= 1\\
R^{12}_{03} &= 2\mathcal{C}_k^2(\alpha,m)\ (e^{-2|\alpha|^2} + \kappa_m e^{-4|\alpha|^2}\cos k\pi)\\
R^{12}_{30} &= 2\mathcal{C}_k^2(\alpha,m)\ (\kappa_m e^{-2|\alpha|^2} + e^{-4|\alpha|^2}\cos k\pi)\\
R^{12}_{11} &= 2\mathcal{C}_k^2(\alpha,m)\ \sqrt{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}\\
R^{12}_{22} &= -2\mathcal{C}_k^2(\alpha,m)\ \sqrt{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}\ e^{-2|\alpha|^2}\cos k\pi\\
R^{12}_{33} &= 2\mathcal{C}_k^2(\alpha,m)\ (\kappa_m e^{-4|\alpha|^2} + e^{-2|\alpha|^2}\cos k\pi)
\end{aligned} \tag{42}$$

Having mapped the bipartite system ($\rho_{12} = \rho_{13}$) into a pair of two qubit presented in the previous subsection, we obtained the final expression of MGQD from a special comparison of the eigenvalues given by the matrix K, which are defined by

$$\lambda^{12}_1 = \frac{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}, \tag{43}$$

$$\lambda^{12}_2 = e^{-4|\alpha|^2}\ \frac{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}, \tag{44}$$

$$\lambda^{12}_3 = e^{-4|\alpha|^2}\ \frac{(1+\kappa_m^2)(1+e^{-4|\alpha|^2}) + 4\kappa_m e^{-2|\alpha|^2}\cos k\pi}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}. \tag{45}$$

$$\rho_{23} = \frac{1}{4}\left[\sigma_0\otimes\sigma_0 + R^{23}_{03}\sigma_0\otimes\sigma_3 + R^{23}_{30}\sigma_3\otimes\sigma_0 + \sum_{k=1,2,3} R^{23}_{kk}\sigma_k\otimes\sigma_k\right] \tag{46}$$

$$\begin{aligned}
R^{23}_{00} &= 1\\
R^{23}_{03} &= 2\mathcal{C}_k^2(\alpha,m)\ (e^{-2|\alpha|^2} + \kappa_m e^{-4|\alpha|^2}\cos k\pi)\\
R^{23}_{30} &= 2\mathcal{C}_k^2(\alpha,m)\ (e^{-2|\alpha|^2} + \kappa_m e^{-4|\alpha|^2}\cos k\pi)\\
R^{23}_{11} &= 2\mathcal{C}_k^2(\alpha,m)\ \sqrt{(1-e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}
\end{aligned}$$

$$R_{22}^{23} = 2\mathcal{C}_k^2(\alpha,m)\ \sqrt{(1-e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}\ \kappa_m e^{-2|\alpha|^2}\cos k\pi$$
$$R_{33}^{23} = 2\mathcal{C}_k^2(\alpha,m)\ (e^{-4|\alpha|^2} + \kappa_m e^{-2|\alpha|^2}\cos k\pi) \tag{47}$$

$$\lambda_1^{23} = \left(\frac{1-e^{-4|\alpha|^2}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right)^2, \tag{48}$$
$$\lambda_2^{23} = \kappa_m^2 e^{-4|\alpha|^2}\left(\frac{1-e^{-4|\alpha|^2}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right)^2, \tag{49}$$
$$\lambda_3^{23} = e^{-4|\alpha|^2}\ \frac{(1+\kappa_m^2)(1+e^{-4|\alpha|^2}) + 4\kappa_m e^{-2|\alpha|^2}\cos k\pi}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}. \tag{50}$$

Now we consider the case of the pure bi-partitioning scheme of type $1|23$. Using the Schmidt decomposition, the state (27) rewrites as

$$|\mathrm{GHZ}_k(\alpha,m)\rangle_{1|23} = \sqrt{\lambda_+}|+\rangle_1 \otimes |+\rangle_{23} + \sqrt{\lambda_-}|-\rangle_1 \otimes |-\rangle_{23} \tag{51}$$

where $|\pm\rangle_1$ (resp. $|\pm\rangle_{23}$) denotes the eigenvectors of the reduced density matrix ρ_1 (resp. ρ_{23} viewed as a single qubit state). The eigenvalues λ_+ and λ_- are given by

$$\lambda_\pm = \frac{1}{2}\left[1 \pm e^{-2|\alpha|^2}\frac{\kappa_m + e^{-2|\alpha|^2}\cos k\pi}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right] \tag{52}$$

The Fano-Bloch representation of the state $\rho_{1|23}$ takes the form

$$\rho_{1|23} = \frac{1}{4}\left[\sigma_0\otimes\sigma_0 + R_{03}^{1|23}\sigma_0\otimes\sigma_3 + R_{30}^{1|23}\sigma_3\otimes\sigma_0 + \sum_{k=1,2,3} R_{kk}^{1|23}\sigma_k\otimes\sigma_k\right] \tag{53}$$

where

$$R_{00}^{1|23} = R_{33}^{1|23} = 1$$
$$R_{03}^{1|23} = R_{30}^{1|23} = e^{-2|\alpha|^2}\frac{\kappa_m + e^{-2|\alpha|^2}\cos k\pi}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}$$
$$R_{11}^{1|23} = -R_{22}^{1|23} = \frac{\sqrt{1-\kappa_m^2 e^{-4|\alpha|^2}}\ \sqrt{1-e^{-8|\alpha|^2}}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi} \tag{54}$$

The eigenvalues

$$\lambda_1^{1|23} = \frac{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-8|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}, \tag{55}$$
$$\lambda_2^{1|23} = \frac{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-8|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}, \tag{56}$$
$$\lambda_3^{1|23} = 2 - \frac{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-8|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}. \tag{57}$$

Limiting cases

The behavior of the pairwise geometric quantum discord is very interesting in antisymmetric states $|\mathrm{GHZ}_1(\alpha, m)\rangle$ in the limiting case $|\alpha| \longrightarrow 0$. In this case, it simply verified that the state (12) reduces to

$$|\mathrm{GHZ}_1(0, m)\rangle = \frac{1}{\sqrt{m+3}}(\sqrt{m+1}|m+1, 0, 0\rangle + |m, 1, 0\rangle + |m, 0, 1\rangle) \tag{58}$$

which coincides with the usual three qubit W states for $m = 0$ [?]. The state (58) is expressed in the Fock-Hilbert basis. In this limit the first qubit is encoded in the Fock states $|m\rangle$ and $|m+1\rangle$ and the two other modes are encoded in the states $|0\rangle$ and $|1\rangle$. In the limit $|\alpha| \longrightarrow 0$, one has $L_m(|\alpha|^2) \simeq 1 - m|\alpha|^2$ and the quantity κ_m (14) writes

$$\kappa_m \simeq 1 - 2m|\alpha|^2. \tag{59}$$

It follows that the eigenvalues (43) reduce, in this limit, to

$$\lambda_1^{12} \longrightarrow 4\,\frac{m+1}{(m+3)^2}, \tag{60}$$

$$\lambda_2^{12} \longrightarrow 4\,\frac{m+1}{(m+3)^2}, \tag{61}$$

$$\lambda_3^{12} \longrightarrow 2\,\frac{m^2+1}{(m+3)^2}. \tag{62}$$

Similarly, for the reduced density matrix ρ_{23} (20) with $k = 1$, it is simple to check that the eigenvalues (48) are given by

$$\lambda_1^{23} \longrightarrow \frac{4}{(m+3)^2}, \tag{63}$$

$$\lambda_2^{23} \longrightarrow \frac{4}{(m+3)^2}, \tag{64}$$

$$\lambda_3^{23} \longrightarrow 2\,\frac{m^2+1}{(m+3)^2}. \tag{65}$$

Finally, for the state $\rho_{1|23}$ (53), viewed as a two qubit system, the corresponding take the following form

$$\lambda_1^{1|23} \longrightarrow 8\,\frac{m+1}{(m+3)^2}, \tag{66}$$

$$\lambda_2^{1|23} \longrightarrow 8\,\frac{m+1}{(m+3)^2}, \tag{67}$$

$$\lambda_3^{1|23} \longrightarrow 2 - 8\,\frac{m+1}{(m+3)^2}, \tag{68}$$

and the pairwise quantum discord is

$$D_{1|23} \longrightarrow 4\,\frac{m+1}{(m+3)^2}, \tag{69}$$

4 Monogamy of geometric quantum discord in quasi-GHZ states

Monogamy of quantum correlations is a property satisfied by certain entanglement measures in a multipartite scenario. Given a tripartite state ρ_{123}, the monogamy condition for a bipartite quantum correlation measure $\mathcal{Q}$ assures that the bipartite quantum correlations in the density operator $\rho_{1|23}$ are distributed in such a way that the following inequality is satisfied

$$\mathcal{Q}(\rho_{1|23}) \geq \mathcal{Q}(\rho_{12}) + \mathcal{Q}(\rho_{13}). \tag{70}$$

Violation of the above inequality will imply that the quantity $\mathcal{Q}$ is polygamous for the corresponding state. Otherwise, this inequality is sufficient for quantum discord to be monogamous.
To illustrate the above analysis, we will investigate the properties of quantum discord monogamy in two different ways (quantum discord and geometric quantum discord using norm 2).

5 Concluding remarks

Summarizing, we have presented in the early of this paper a class of the single mode excited entangled coherent states (SMEECSs)$|\psi_p(\alpha, m)\rangle$, which are obtained through actions of creation operator on the entangled coherent states. Then, we have exhibited the important properties of quantum entanglement by using different ways (specially, the concurrence, quantum discord and its version geometric). The first way, we have studied the concurrence for bipartite systems and investigated the influence of phonon excitations numbers on quantum entanglement. We also employed the other process for studied the quantum correlation of add coherent states for tripartite quantum states (see the equations (**??**) and (**??**) by the quantum discord. Thus, we found two explicit analytic expressions of this measure and the results obtained are discussed. Another way which treated the quantum correlations by introducing the geometric version of quantum discord, at this stage we derived a necessary and sufficient condition. Specially, for the case of three-qubit states we have proposed in the our discussion, two version(i.e.symmetric, antisymmetric states, respectively) and the result obtained is explained in terms of different number photon excitations(i.e.the influence of m on geometric quantum discord). To close our work, we have employed the concept of quantum monogamy corresponding to quantum discord and its geometric version. In particular, we have investigated the relation between discord monogamy and a genuine tripartite entanglement measure for three-qubit pure states. Therefor, We have demonstrated that the quantum correlations examined by the entropic measure, geometric measure respectively does not satisfy the monogamy relation(70). A very important result is derived in this work, from a value determined of $|\alpha|^2$, we see that no effect of the addition of the photon can be found on the measurement of monogamy. The analysis presented in this letter can be extended to the effect of subtracting the photon of tripartite GHZ coherent states.

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