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PhD TUTORIAL

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Abstract. Nonclassical states of light are of fundamental importance in quantum optics. The properties of these states are due to the quantal nature of the electromagnetic field. In recent years there have been many experimental demonstrations of the realizability of nonclassical states in various physical schemes such as resonance fluorescence, four-wave mixing, centre-of-mass motion of a trapped ion and manipulation of field in a cavity. It is of interest to introduce new classes of nonclassical states and investigate their properties. In this context, generalization of the concept of coherent states (CSs) has played a major role. New concepts such as the nonlinear coherent states (NCSs) and interference in phase space have emerged from such studies. In this tutorial review we are concerned with the construction of new classes of nonclassical states by generalizing the notion of CSs. Our investigations on the photon-added coherent states (PACSs) indicate that these states can be interpreted as NCSs. Also, we introduce a new class of nonclassical states related to the PACSs. Having introduced a realizable example of NCSs, we extend the notion of even and odd CSs to the case of NCSs by introducing even and odd NCSs. With this new definition we interrelate some of the well known states of light. We suggest a scheme to generate a class of even and odd NCSs in the centre-of-mass motion of a trapped, laser-cooled, two-level ion.

Keywords: Nonlinear coherent states, squeezed states, photon statistics, raising operators, even coherent states, odd coherent states

1. Introduction

An important concept which emerges from the study of the quantum harmonic oscillator is the notion of coherent states (CSs), introduced by Schrödinger [1] as wavepackets whose dynamics resembles that of a classical particle in a quadratic potential. These states are useful in various branches of physics [2, 3]. The notion of CSs has been generalized in very many ways. Motivations to generalize the concept have arisen from symmetry considerations, algebraic aspects and dynamics. Generalization based on symmetry considerations has led to defining CSs for arbitrary Lie groups [4, 6]. CSs for the deformed algebras have been introduced by extending the algebraic definition [7, 8]. Based on dynamics, CSs have been constructed for systems other than the harmonic oscillator [9]. This tutorial article is organized as follows. After a brief review of various CSs known in the literature, nonlinear coherent states (NCSs), which can be classified as an algebraic generalization of CSs, are introduced.

1.1. Coherent states of the harmonic oscillator

The harmonic oscillator is a well studied system in both classical and quantum physics. In quantum physics the description of the harmonic oscillator is most elegantly achieved in terms of creation and annihilation operators. These operators arise naturally in the process of factorizing the harmonic oscillator Hamiltonian

$$\hat{H} = \tfrac{1}{2}(\hat{p}^2 + \hat{x}^2), \tag{1}$$

in which $\hat{x}$ and $\hat{p}$ are the position and momentum operators respectively. Here the mass and frequency of the oscillator are set equal to unity. The operators $\hat{x}$ and $\hat{p}$ satisfy the commutation relation $[\hat{x}, \hat{p}] = \mathrm{i}$. We have set $\hbar = 1$. Defining two non-Hermitian operators,

$$\hat{a} = \frac{\hat{x} + \mathrm{i}\hat{p}}{\sqrt{2}} \tag{2}$$

and its conjugate

$$\hat{a}^\dagger = \frac{\hat{x} - \mathrm{i}\hat{p}}{\sqrt{2}}, \tag{3}$$

which satisfy $[\hat{a}, \hat{a}^\dagger] = 1$, the Hamiltonian of the oscillator can be written as

$$\hat{H} = \hat{a}^\dagger\hat{a} + \tfrac{1}{2}. \tag{4}$$

The eigenstates of $\hat{H}$ are represented as $|n\rangle$, $n = 0, 1, 2, \ldots$. The action of $\hat{a}$ and $\hat{a}^\dagger$ on the state $|n\rangle$ is given by

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{5}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{6}$$

with the condition that $\hat{a}$ annihilates the ground state $|0\rangle$. The operators $\hat{a}$ and $\hat{a}^\dagger$ are respectively the annihilation and creation operators because of the way they act on the number states.

Determining the form of a wavepacket whose dynamics resembles that of a classical particle in a harmonic oscillator potential is equivalent to determining the eigenstates of the annihilation operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [10]. These eigenstates are the CSs for the harmonic oscillator. The number state expansion for the CSs, normalized to unity, is

$$|\alpha\rangle = \exp\left(\frac{-|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \qquad \alpha \in \mathcal{C}. \tag{7}$$

Using the unitary operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, the CSs defined in equation (7) can be obtained as

$$|\alpha\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle. \tag{8}$$

The uncertainty in position $\hat{x}$ is defined as

$$(\triangle x) = \sqrt{\langle\hat{x}^2\rangle - \langle x\rangle^2} \tag{9}$$

and a similar definition holds for the momentum. For the harmonic oscillator CSs the uncertainties in $\hat{x}$ and $\hat{p}$ are equal to $\sqrt{1/2}$. The product $(\triangle x)(\triangle p)$ is $1/2$ in units of $\hbar$, the minimum value allowed by the Heisenberg uncertainty relation for any state. Hence, these states are minimum uncertainty states (MUSs).

The CSs are defined in three ways:

(1) the eigenstate of $\hat{a}$, which is the algebraic definition,
(2) the unitarily deformed vacuum state (refer to equation (8)), which is the group theoretic definition, and
(3) the MUS.

In the case of the simple harmonic oscillator the three definitions are equivalent.

1.2. Hilbert space properties of coherent states

Some of the important features of the CSs [11] are the following.

- Two CSs, say, $|\alpha\rangle$ and $|\beta\rangle$, are not orthogonal to each other:
$$|\langle\beta|\alpha\rangle|^2 = \exp(|\alpha - \beta|^2). \tag{10}$$
- The most important property of the CSs is that they form an overcomplete set:
$$\frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha| = 1. \tag{11}$$
The integration measure $d^2\alpha$ represents the area element $d(\mathrm{Re}\,\alpha)\,d(\mathrm{Im}\,\alpha)$ and the integration is performed over the entire complex plane.
- An arbitrary density operator $\hat{\rho}$ can be expanded in terms of CS projections as
$$\hat{\rho} = \frac{1}{\pi}\int d^2\alpha\ P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|. \tag{12}$$
This expression gives the diagonal representation for the operator $\hat{\rho}$ in the CS basis. The function $P(\alpha, \alpha^*)$ is called the Glauber–Sudarshan or P function [12].
- As a consequence of the diagonal representation, the expectation value of a normally ordered operator $\hat{O}(\hat{a}, \hat{a}^\dagger)$, in which all the annihilation operators are to the right of the creation operators, is given by
$$\langle\hat{O}\rangle = \int d^2\alpha\ P(\alpha, \alpha^*)O(\alpha, \alpha^*). \tag{13}$$
Here $\langle\cdots\rangle$ represents the expectation value in an arbitrary state and $P(\alpha, \alpha^*)$ is the P function for that state.
- The number distribution $|\langle n|\alpha\rangle|^2$ for the CSs is
$$|\alpha|^2 = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \tag{14}$$
This is a Poisson distribution whose mean and variance are equal to $|\alpha|^2$.

1.3. Uses of coherent state representation

The diagonal representation, equation (12), provided by the CSs is of great use in calculating various physical quantities of interest.

- The expectation value of the operator $(\triangle\hat{x})^2 = (\hat{x} - \langle x\rangle)^2$ can be written, after normal ordering, as
$$\langle(\triangle\hat{x})^2\rangle = \tfrac{1}{2} + \int d^2\alpha\ P(\alpha, \alpha^*)(|\alpha|^2 - \langle\alpha\rangle - \langle\alpha^*\rangle)^2. \tag{15}$$
Here equations (2) and (3) have been used to write $\hat{x}$ as $(\hat{a}^\dagger + \hat{a})/\sqrt{2}$. If the value of $\langle(\triangle\hat{x})^2\rangle$ is less than $1/2$ for some state, then equation (15) would mean that $P(\alpha, \alpha^*)$ cannot be positive definite everywhere on the complex plane. For such states the P function ceases to be a meaningful probability density function. If the uncertainty in x for a state is less than $1/2$ (in the units of $\hbar$) the state is said to be 'squeezed' in x. A similar argument holds for p.
- For a Poisson distribution the mean is equal to its variance. If the variance of a distribution is less (respectively, greater) than that of its mean, the distribution is said to be sub-Poissonian (respectively, super-Poissonian). Sub-Poissonian statistics for the number distribution of a state means that the corresponding P function is not positive definite [13].
- Yet another property of a state that can make the underlying P function nonpositive is an oscillatory number distribution, i.e. the distribution vanishes for finite values of n [14].

Squeezing, sub-Poissonian statistics and oscillatory number distribution are called nonclassical features as the P function ceases to be a classical probability density for the states exhibiting any of those features. A state with any of the above features is said to be nonclassical.

1.4. Coherent states of the electromagnetic field

In the process of quantizing the free electromagnetic field one is led to introduce operators which have the same commutation relation as the usual creation and annihilation operators. As a result of quantization, each mode of the field can be regarded as an independent harmonic oscillator. An arbitrary field distribution will have many modes and the Hamiltonian for the field can be written as

$$\hat{H} = (\tfrac{1}{2}) \sum_{k=0}^{\infty} (\hat{a}_k^{\dagger} \hat{a}_k + \hat{a}_k \hat{a}_k^{\dagger}) \omega_k. \tag{16}$$

Each term in the summation corresponds to one mode of the field. The operators $\hat{a}_k$ and $\hat{a}_k^{\dagger}$ satisfy $[\hat{a}_k, \hat{a}_{k'}^{\dagger}] = \delta_{kk'}$. Here $\hat{a}_k$ and $\hat{a}_k^{\dagger}$ are the annihilation and creation operators respectively, for the oscillator corresponding to the kth mode, and ω_k is the frequency of the mode

In the context of electromagnetic fields, CSs were introduced by Glauber [15, 16] as the eigenstates of the annihilation operator $\hat{a}$. These CSs have all the Hilbert space properties of the harmonic oscillator CS. In particular, the diagonal representation establishes the formal equivalence of classical and quantal theories of photon-counting distribution [12]. In the classical theory of radiation the probability that m counts are registered in a time interval T is given by

$$P(m, T) = \int_0^{\infty} \frac{(\gamma I T)^m}{m!} \exp(-\gamma I T)\, \mathrm{d}I. \tag{17}$$

In the above expression γ is a parameter which depends on the detector sensitivity, spectral characteristics of the incident radiation etc and I is the intensity of the incident radiation. In quantum theoretic formulation the counting distribution is given by

$$P(m, T) = \left\langle : \frac{(\gamma T \hat{a}^{\dagger} \hat{a})^m}{m!} \exp(-\gamma T \hat{a}^{\dagger} \hat{a}) : \right\rangle. \tag{18}$$

Here : .. : indicates that the operator is normally ordered. This definition for $P(m, T)$ assumes that the detection of photons is by absorption. Using the diagonal representation, this expression can be written as

$$P(m, T) = \int \frac{(\gamma T |\alpha|^2)^m}{m!} \exp(-\gamma T |\alpha|^2)\, \mathrm{d}^2\alpha. \tag{19}$$

The quantal expression for the photon-counting distribution equation (19) has a structure similar to the classical case equation (17) that is based on analytic signal representation.

1.5. Coherent states for general potential

The notion of CSs can be generalized to systems other than the harmonic oscillator [9, 17]. The motivation is to construct CSs for one-dimensional potentials with *unequally* spaced energy levels. The construction is such that the resultant states are localized, follow the classical motion and disperse as little as possible in time. The idea is to rewrite the Hamiltonian, for a particle moving in an arbitrary potential, in the form of a harmonic oscillator Hamiltonian. Let $V(x)$ be a one-dimensional, local potential with one confining region. The classical Hamiltonian for a particle of mass m in such a potential is

$$H = \frac{p^2}{2m} + V(x). \tag{20}$$

Here x and p are canonically conjugate to each other. The bounded motion of the particle with total energy E is periodic with a period T given by

$$T = \sqrt{\frac{m}{2}} \oint \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \tag{21}$$

The integration is carried out around the branch cut separating the two turning points. The frequency of oscillation is $\omega = (2\pi)T^{-1}$. Defining two new variables

$$X = A \sin(\omega t) \tag{22}$$

and

$$\begin{aligned} P &= m \frac{\mathrm{d}}{\mathrm{d}t} X, \\ &= p \frac{\mathrm{d}}{\mathrm{d}x} X \end{aligned} \tag{23}$$

the Hamiltonian given by equation (20) can be rewritten as

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}, \tag{24}$$

$$= \tfrac{1}{2} m \omega^2 A^2. \tag{25}$$

Here A is the amplitude of X. The frequency ω and the amplitude A are functions of energy. The Poisson bracket of the new variables X and P is not unity and hence the transformation from (x, p) to (X, P) is noncanonical. However, the advantage of the transformation is that any Hamiltonian can be rewritten in the form of an oscillator Hamiltonian in terms of the noncanonical variables X and P.

The operators corresponding to the observables X and P can be written as

$$\hat{X} = X, \tag{26}$$

$$\hat{P} = \frac{-\mathrm{i}\hbar}{2} \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{d}X}{\mathrm{d}x} + \frac{\mathrm{d}X}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}x} \right]. \tag{27}$$

The commutation between the two operators becomes

$$[\hat{X}, \hat{P}] = \mathrm{i}\hbar \left[\frac{\mathrm{d}X}{\mathrm{d}x} \right]^2 \tag{28}$$

and the corresponding uncertainty relation is

$$\langle (\triangle \hat{X})^2 \rangle \langle (\triangle \hat{P})^2 \rangle \Bigg/ \left\langle \left[\frac{\mathrm{d}X}{\mathrm{d}x} \right]^2 \right\rangle \geqslant \hbar^2/4\,. \tag{29}$$

The CSs for the potential $V(x)$ are defined as the states which minimize the uncertainty relation equation (29). This definition implies that these states, denoted as $|X, P\rangle$, satisfy

$$[\hat{X} + \mathrm{i}\hat{P}]|X, P\rangle = (\langle X \rangle + \mathrm{i}\langle P \rangle)|X, P\rangle. \tag{30}$$

The symbol $\langle \cdots \rangle$ represents the expectation value in the state$|X, P\rangle$. Making use of the above definition, CSs have been constructed for the Poschl–Teller potential and harmonic oscillator with a centripetal barrier [18]. The procedure indicated above to construct CSs for confining potentials can be extended to potentials which have both a discrete and continuous spectrum. The Rosen–Morse and Morse potentials are examples of such nonconfining potentials and their respective CSs are known [19].

1.6. Superposition of coherent states

The CSs of the harmonic oscillator are special in the sense that only for these states is the P-function a delta function. For any other pure state the P-function is more singular than a delta function. Thus the CSs are the ones closest to a classical state, i.e. the phase-space distribution is well localized in both the position and momentum variables. However, when two such classical states are superposed the resultant states exhibit many nonclassical features. An important case is the superposition of two CSs of same amplitude with their phases differing by π. The symmetric combination $|\alpha\rangle + |-\alpha\rangle$ is

$$|\alpha, +\rangle = \frac{1}{\sqrt{(\cosh(|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle, \qquad \alpha \in \mathcal{C}. \tag{31}$$

The state $|\alpha, +\rangle$ is called an even coherent state (ECS). The antisymmetric superposition involves only the odd number states and is given by

$$|\alpha, -\rangle = \frac{1}{\sqrt{(\sinh(|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle, \qquad \alpha \in \mathcal{C}. \tag{32}$$

This antisymmetric superposition of $|\alpha\rangle$ and $|-\alpha\rangle$ is called an odd coherent state (OCS). The even and odd CSs and their relation to the $SU(1,1)$ group is clarified in [20].

1.7. Nonlinear coherent states

Another class of CSs can be introduced by deforming the basic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. A well studied system is the q-deformed oscillator, where the basic commutation relation is taken to be $\hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = q^{-\hat{n}}$, in which $\hat{n}$ is the number operator satisfying $[\hat{n}, \hat{a}] = -\hat{a}$ and $[\hat{n}, \hat{a}^\dagger] = -\hat{a}^\dagger$. This relation is said to be deformed as it becomes the usual commutation relation when $q = 1$. The q-deformation was originally introduced to integrate the Yang–Baxter equation for the inverse quantum scattering [21–23]. In the context of the harmonic oscillators the q-deformation was introduced [7, 8] by deforming the $SU(2)$ algebra.

The understanding that deformation amounts to introducing nonlinearity is due to Manko *et al* [24]. Define two new operators $\hat{A}$ and $\hat{A}^\dagger$ by deforming the annihilation and creation operators, that is

$$\hat{A} = f(\hat{n})\hat{a} \qquad \text{and} \qquad \hat{A}^\dagger = \hat{a}^\dagger f(\hat{n}). \tag{33}$$

The operator-valued function $f(\hat{n})$ is the deforming function. Set $f(\hat{n}) = \sqrt{\frac{\sinh(\lambda\hat{n})}{\sinh(\lambda)\hat{n}}}$ with the parameter λ taken to be real. If λ is defined in terms of another real number q through the equation $q = \exp(\lambda)$, then the operators of equation (33) will satisfy the commutation relation

$$\hat{A}\hat{A}^\dagger - q\hat{A}^\dagger\hat{A} = q^{-\hat{a}^\dagger\hat{a}}. \tag{34}$$

The transformation $\hat{a} \rightarrow \hat{A}$ is noncanonical as the commutator $[\hat{A}, \hat{A}^\dagger] \neq 1$.

To bring out the physical meaning of the above transformation it is better to look at the underlying classical system. The classical oscillator is described in terms of the complex amplitudes $\alpha = (x + \mathrm{i}p)/\sqrt{2}$ and its conjugate α^*. Their Poisson bracket, with respect to q and p, is $\{\alpha, \alpha^*\} = -\mathrm{i}$ and the Hamiltonian for the oscillator becomes $\alpha\alpha^*$. To describe the dynamics of the oscillator, or for that matter any system, it is required to specify the Hamiltonian in a suitable set of variables, and the Poisson bracket among those variables. Systems other than the harmonic oscillator are to be described by different Hamiltonians. Equivalently, the Poisson bracket can be changed by making a noncanonical transformation while preserving the form of the Hamiltonian to be that of the harmonic oscillator. Such a prescription would then describe a different physical system.

As an example consider the classical q-oscillator, which is defined in terms of two new variables

$$\beta = \left[\sqrt{\frac{\sinh(\lambda\alpha\alpha^*)}{\sinh(\lambda)\alpha\alpha^*}}\right]\alpha \tag{35}$$

and its conjugate β^*. The Hamiltonian is taken to be $\beta\beta^*$. The transformation defined by equation (35) is noncanonical as $\{\beta, \beta^*\} = -\mathrm{i}\frac{\lambda}{\sinh(\lambda)}\sqrt{1 + \sinh^2(|\alpha|^2)}$. In terms of the original variables α and α^*, the Hamiltonian becomes $\sinh(\lambda\alpha\alpha^*)/\sinh(\lambda)$. The time evolution equation for the variable β is

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta = \{\beta, H\} \tag{36}$$

and its solution is

$$\beta(t) = \beta(0) \exp\left[-\mathrm{i}t \frac{\lambda}{\sinh(\lambda)} \cosh(\lambda\alpha\alpha^*)\right]. \tag{37}$$

The quantity $\alpha\alpha^*$ is a constant of motion and hence the frequency of oscillation of β in equation (37) can be identified as

$$\omega = \frac{\lambda}{\sinh(\lambda)} \cosh(\lambda\alpha\alpha^*). \tag{38}$$

On applying the canonical quantization rule, $\alpha \rightarrow \hat{a}$, to equation (35) the variable β becomes the operator $\hat{A}$ defined in equation (33). The analysis of the underlying classical system shows that q-deformation amounts to making the frequency dependent on energy as given by equation (38). In the limit $\lambda \rightarrow 0$ the frequency $\omega \rightarrow 1$. This is to be expected as the q-commutation relation becomes the usual relation $\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$ in the above limit.

So, the essential feature of deformation is that it is associated with energy dependent frequency provided the deforming function depends on the number operator. The nonlinearity depends on the the form of $f(\hat{n})$. Denoting an arbitrarily deformed $\hat{a}$ by $\hat{B}$, f-oscillators are defined by the Hamiltonian

$$H_f = \tfrac{1}{2}[\hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}]. \tag{39}$$

The suffix f indicates that the operators are f-deformed. The eigenstates of this Hamiltonian are the same as those of the usual oscillator as the deforming function is a function of the number operator. The energy-dependent frequency in the classical case is seen as eigenvalues which are nonlinear functions of n in the quantum case.

Although introduced as a mathematical generalization of the basic commutation relation, the q-deformed oscillators have found application in a large number of realistic physical systems such as polyatomic molecules and matter–radiation interaction. The potential energy between the atoms of a polyatomic molecule has anharmonic terms. Since the deformed oscillators can be interpreted as anharmonic oscillators [25], they are suitable to model the vibrations of polyatomic molecules [26, 27].

The known physical models when subjected to q-deformation predict results which are different from the undeformed ones. This makes it possible to verify experimentally the relevance of deformed algebras for physics. Such studies have been carried out in the context of determining form-factors for electron–photon scattering [28], the effect of deformation on interference [29] etc. A more realistic and experimentally verifiable consequence of q-deformation has been suggested in the context of the interaction of a two-level atom with an electromagnetic field. The interaction can be described by the model due to Jaynes and Cummings [30], in which the atom is treated as a dipole and the field is treated as an oscillator. An extension of this model considers intensity-dependent coupling between the field and the atom [31, 32]. The q-analogue of this nonlinear interaction model predicts a 'revivals and collapses' phenomenon which is quantitatively different from what is expected from the undeformed model [33]. The revival time becomes shorter with increasing value of q. This result can be verified experimentally.

A nonlinear oscillator whose frequency depends on energy provides the physical motivation to introduce deformed annihilation operators. CSs for such systems are defined in [34] as the eigenstates of $\hat{B}$. These states, denoted as $|\alpha, f\rangle$, obey

$$\hat{B}|\alpha, f\rangle = \alpha|\alpha, f\rangle. \tag{40}$$

These eigenstates are called NCSs. Expanding the states in the number state basis as

$$|\alpha, f\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{41}$$

and substituting in equation (40) yields

$$c_n = \frac{\alpha^n}{\sqrt{n!}(f(n))!}c_0, \tag{42}$$

in which $(f(n))! = f(0)f(1)f(2), \ldots, f(n)$. Normalization of the state $|\alpha, f\rangle$ determines the value of $|c_0|$ as

$$|c_0|^{-2} = \sum_{n=0}^{\infty} \frac{|\alpha|^2}{n![(f(n))!]^2}. \tag{43}$$

The constant c_0 should satisfy $0 < |c_0| < \infty$. This implies

$$|\alpha| \leqslant \lim_{n\to\infty} n[f(n)]^2. \tag{44}$$

Depending on the form of the deforming function, the range of α may be restricted to a finite disc on the complex plane. If $f(\hat{n})$ vanishes for any value of n other than $n = 0$, then the deformed annihilation operator becomes nilpotent. That is, the deformed annihilation operator raised to some positive power vanishes identically. If N is the maximum value of n for which $f(n)$ vanishes, the NCS is constructed on the Hilbert space spanned by the set $|N\rangle, |N+1\rangle, \ldots$.

An example of an NCS which is defined for finite values of α is the harmonious state [35]. These are the eigenstates of an operator $\hat{b}$ whose action on the number state is given by

$$\hat{b}|n\rangle = |n-1\rangle, \qquad n = 1, 2, 3, \ldots \tag{45}$$

and it annihilates the vacuum state $|0\rangle$. These eigenstates are NCSs corresponding to the deforming function $1/\sqrt{1+\hat{a}^\dagger\hat{a}}$. Using this form for $f(n)$ in equation (44) implies that the harmonious states are defined on the disc $|\alpha| \leqslant 1$. The usual CS $|\alpha\rangle$ and q-CS [36] are defined for all values of α.

Similar to the states $|\alpha\rangle$, the NCSs have the following Hilbert space properties.

- The scalar product between two NCSs corresponding to two different values of the eigenvalue is

$$\langle\alpha, f|\beta, f\rangle = N_\alpha N_\beta \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n![(f(n))!]^2}. \tag{46}$$

  Here N_α and N_β are the normalization constants for the states $|\alpha, f\rangle$ and $|\beta, f\rangle$ respectively.
- The resolution of identity can be written as

$$\int \mathrm{d}\mu\,(\alpha)|\alpha, f\rangle\langle\alpha, f| = 1. \tag{47}$$

  Here $\mathrm{d}\mu$ is a suitable measure to be determined. Setting $\alpha = \rho\exp(\mathrm{i}\theta)$ and substituting the number state expansion for $|\alpha, f\rangle$ gives

$$\int_0^{\rho'} \rho^{2n+1}|c_0|^2\mu(\rho)\,\mathrm{d}\rho = n!([f(n)]!)^2. \tag{48}$$

  These are the moment equations for the measure μ. If a function $\mu(\rho)$ exists which satisfies the above equation for all n, then the identity can be resolved in terms of $|\alpha, f\rangle$. The integration limit ρ' is the maximum value of $|\alpha|$, as restricted by equation (44), for which the states $|\alpha, f\rangle$ converge.

Eigenstates of generalized annihilation operators have been introduced in various other contexts: generalized bosonic operators which satisfy the Heisenberg–Weyl algebra and their CSs [37–39], squeezing [40]; factorization of Hamiltonian and isospectral oscillators [41, 42] and description of the stationary states of a trapped, sideband-cooled two-level atom [43].

In section 2 of this paper we introduce a physically realizable example of NCSs, namely, the photon-added coherent states (PACSs), and extend its definition further to construct a new of class of NCSs. Section 3 of the paper deals with the extension of the notion of NCSs to the case of even and odd CSs, leading to the definition of even and odd NCSs. As a consequence of the definition we establish that the squeezed vacuum and the squeezed first excited states can be interpreted as even and odd NCSs respectively. In section 4 we discuss the possibility of generating a class of even and odd NCSs in the interaction of a harmonically trapped two-level ion interacting with two external laser fields of suitable frequencies.

2. Photon-added coherent states as nonlinear coherent states

PACSs $|\alpha, m\rangle$ are defined [44] as

$$|\alpha, m\rangle = \frac{\hat{a}^{\dagger m}|\alpha\rangle}{\sqrt{\langle\alpha|\hat{a}^m\hat{a}^{\dagger m}|\alpha\rangle}}, \tag{49}$$

where m is a non-negative integer. The states $|\alpha, m\rangle$ exhibit nonclassical features such as phase squeezing and sub-Poissonian statistics. These states are produced in the interaction of a two-level atom, having a ground state $|\mathrm{g}\rangle$ and an excited state $|\mathrm{e}\rangle$, with a single-mode cavity field. The Hamiltonian which describes the atom–field interaction, in the rotating wave approximation (RWA), can be written as

$$\hat{H}_{\mathrm{int}} = \hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger). \tag{50}$$

Here $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively for the field in the cavity. The operator $\hat{\sigma}_+$ is the flip operator corresponding to the atomic transition $|\mathrm{g}\rangle \to |\mathrm{e}\rangle$ and its conjugate $\hat{\sigma}_-$ corresponds to the transition $|\mathrm{e}\rangle \to |\mathrm{g}\rangle$. The operator $\hat{\sigma}_+$ (respectively, $\hat{\sigma}_-$) annihilates the state $|\mathrm{e}\rangle$ (respectively, $|\mathrm{g}\rangle$). Let the initial state of the atom + field system be $|\alpha\rangle|\mathrm{e}\rangle$, where $|\alpha\rangle$ is a CS of the cavity field. The state $|\psi(t)\rangle$ of the system at a later time t is given by

$$|\psi(t)\rangle = \exp(-\mathrm{i}\hat{H}_{\mathrm{int}}t/\hbar)|\alpha\rangle|\mathrm{e}\rangle. \tag{51}$$

If the coupling constant g is small, the state of the atom–field at time t, such that $gt \ll 1$, can be written as

$$|\psi(t)\rangle \simeq |\alpha\rangle|\mathrm{e}\rangle - \frac{\mathrm{i}\hat{H}_{\mathrm{int}}t}{\hbar}|\alpha\rangle|\mathrm{e}\rangle. \tag{52}$$

In writing the above equation we have expanded the exponential in the RHS of equation (51) and retained only the terms which are first order in gt. On using equation (50) for the interaction Hamiltonian, the state $|\psi(t)\rangle$ becomes $|\alpha\rangle|\mathrm{e}\rangle - \mathrm{i}gt\hat{a}^\dagger|\alpha\rangle|\mathrm{g}\rangle$. If the atom, on its exit from the cavity, is detected to be in the ground state $|\mathrm{g}\rangle$, then the cavity field is in the state $\hat{a}^\dagger|\alpha\rangle$. This state has to be normalized as it is has been obtained after truncating the unitary time-evolution operator in equation (51) to first order in gt. If the interaction is a multiphoton process, $\hat{a}(\hat{a}^\dagger) \to \hat{a}^m(\hat{a}^{\dagger m})$, the cavity field will be produced in a state proportional to $\hat{a}^{\dagger m}|\alpha\rangle$. It is also possible to produce such states in the conditional measurement on the output of a beam-splitter [45].

To show that PACSs are NCSs [46] we begin with the definition of CSs. The states $|\alpha\rangle$ satisfy, by definition,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{53}$$

Premultiplying both sides of this equation by $\hat{a}^{\dagger m}$ (m is a non-negative integer) and using the commutation relation $[\hat{a}, \hat{a}^{\dagger m}] = m\hat{a}^{\dagger m-1}$, the above equation is written as

$$(\hat{a}\hat{a}^{\dagger m} - m\hat{a}^{\dagger(m-1)})|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle. \tag{54}$$

Using identity $\frac{1}{1+\hat{a}^\dagger\hat{a}}\hat{a}\hat{a}^\dagger = 1$ leads to

$$\left(\hat{a} - \frac{m}{1+\hat{a}^\dagger\hat{a}}\hat{a}\right)\hat{a}^{\dagger m}|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle. \tag{55}$$

Equation (55) gives the expression for $f(\hat{n}, m)$ as

$$f(\hat{n}, m) = 1 - \frac{m}{1+\hat{a}^\dagger\hat{a}}. \tag{56}$$

This shows that the PACSs can be interpreted as NCSs with the deformation function $1 - m/(1+\hat{a}^\dagger\hat{a})$. A more general result is that the action of $\hat{a}^\dagger$ on any NCS is again an NCS [50].

2.1. $|\alpha, m\rangle$ as deformed number state

The PACSs are the eigenstates of $f(\hat{n}, m)\hat{a}$ with $f(\hat{n}, m)$ given by equation (56). The operator $f(\hat{n}, m)\hat{a}$ annihilates the vacuum state $|0\rangle$ and the m-photon state $|m\rangle$. The states in between the vacuum and the m-photon states are not annihilated by this operator. In this sense it is different from the m-photon annihilation operator $\hat{a}^m$ which annihilates all the number states $|i\rangle$, $i = 0, 1, 2, \ldots, m$. To write $|\alpha, m\rangle$ as a nonunitarily deformed number state, let

$$\hat{B} = \left(1 - \frac{m}{1+\hat{a}^\dagger\hat{a}}\right)\hat{a}. \tag{57}$$

The adjoint of $\hat{B}$ is given by

$$\hat{B}^\dagger = \hat{a}^\dagger\left(1 - \frac{m}{1+\hat{a}^\dagger\hat{a}}\right). \tag{58}$$

A sector S_0 is constructed in the harmonic oscillator Hilbert space $\mathcal{H}$ by repeatedly applying $\hat{B}^\dagger$ on the vacuum state $|0\rangle$. The sector S_0 is the space spanned by the Fock states $\{|i\rangle, i = 0, 1, 2, \ldots, m-1\}$ and it is finite dimensional. Starting with $|m\rangle$, also annihilated by $\hat{B}$, we construct another sector S_m in $\mathcal{H}$ by the repeated application of $\hat{B}^\dagger$ on it. The sector S_m is the set $\{|i\rangle, i = m, m+1, \ldots\}$ and it is of infinite dimension. Using the method given in the appendix, we construct an operator $\hat{G}^\dagger$ such that $[\hat{B}, \hat{G}^\dagger] = 1$ holds in the sector S_m. To carry out the construction we set $p = 1$ and $j = m$ in equation (135) and this yields

$$\hat{G}^\dagger = \hat{a}^\dagger. \tag{59}$$

Thus, on the sector S_m we have $[\hat{B}, \hat{a}^\dagger] = 1$. Hence the PACSs, which are the eigenstates of $\hat{A}$ given in equation (57), can be written as $\exp(\alpha\hat{a}^\dagger)|m\rangle$ but for a multiplicative normalization constant. However, this is not a unitary deformation since $\exp(\alpha\hat{a}^\dagger)\exp(\alpha^*\hat{a}) \neq 1$.

2.2. Eigenstates of $f(\hat{n}, m)\hat{a}$ with negative m

The definition of $f(\hat{n}, m)$, given by equation (56) can be extended to include negative values for m.

Expression for $f(\hat{n}, m)$. Denoting the NCSs corresponding to negative m by $|\alpha, -m\rangle$, the equation to determine them is

$$\left(1 + \frac{m}{1+\hat{a}^\dagger\hat{a}}\right)\hat{a}|\alpha, -m\rangle = \alpha|\alpha, -m\rangle. \tag{60}$$

The normalized $|\alpha, -m\rangle$ is given by

$$\begin{aligned} |\alpha, -m\rangle &= Nm!\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}(n+1)(n+2)\cdots(n+m)}|n\rangle; \\ N^{-1} &= m!\sqrt{\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}n!}{(n+m)!^2}} \\ &= \sqrt{{}_2F_2(1, 1, m+1, m+1, |\alpha|^2)}. \end{aligned} \tag{61}$$

Here ${}_2F_2(1, 1, m+1, m+1, |\alpha|^2)$ is the generalized hypergeometric function [47]. Setting $f(n) = 1 + m/(1+n)$ in equation (44), we can infer that the states are normalizable for all values of α.

The number state expansion for the state $|\alpha, m\rangle$ [44] is

$$|\alpha, m\rangle = \frac{\exp(-|\alpha|^2/2)}{\sqrt{L_m(-|\alpha|^2)m!}} \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(m+n)!}}{n!} |n+m\rangle, \quad (62)$$

where $L_m(x)$ is a Laguerre polynomial of order m defined by

$$L_m(x) = \sum_{n=0}^{m} \frac{(-x)^n m!}{(n!)^2 (m-n)!}. \quad (63)$$

The state $|\alpha, -m\rangle$ given by equation (61) involves a superposition of all the Fock states starting with the vacuum state $|0\rangle$. In the number state expansion of $|\alpha, m\rangle$ the states $|0\rangle, |1\rangle \cdots |m-1\rangle$ are not present. This important difference leads to different limiting cases of the states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ as $\alpha \to 0$. In the limit $\alpha \to 0$ the state $|\alpha, -m\rangle$ becomes the vacuum state $|0\rangle$ irrespective of the value of m and the state $|\alpha, m\rangle$ becomes the number state $|m\rangle$. In the limit $m \to 0$ the states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ become $|\alpha\rangle$. Thus, $|\alpha, -m\rangle$ (respectively, $|\alpha, m\rangle$) is a state that is intermediate between the vacuum state (respectively, the number state $|m\rangle$) and the CS.

The PACSs are obtained by the action of $\hat{a}^{\dagger m}$ on $|\alpha\rangle$. The states $|\alpha, -m\rangle$ can be written in a similar form using the inverse operators $\hat{a}^{-1}$ and $\hat{a}^{\dagger -1}$ [48]. These operators are defined in terms of their action on the number state $|n\rangle$ as follows:

$$\hat{a}^{-1}|n\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle, \quad (64)$$

$$\hat{a}^{\dagger -1}|n\rangle = \frac{1}{\sqrt{n}}|n-1\rangle \qquad \text{for} \quad n \neq 0, \quad (65)$$

$$\hat{a}^{\dagger -1}|0\rangle = 0. \quad (66)$$

The operator $\hat{a}^{-1}$ is the right inverse of $\hat{a}$ and $\hat{a}^{\dagger -1}$ is the left inverse of $\hat{a}^\dagger$. Using these inverse operators and equation (61) the state $|\alpha, -m\rangle$ is written as

$$|\alpha, -m\rangle = N\hat{a}^{\dagger -m}\hat{a}^{-m}|\alpha\rangle. \quad (67)$$

The states $|\alpha, -m\rangle$ correspond to the NCSs with $-m$ replacing m in $f(\hat{n}, m)$. However, they are obtained by the action of $\hat{a}^{\dagger -m}\hat{a}^{-m}$ on $|\alpha\rangle$ and not $\hat{a}^{\dagger -m}$ on $|\alpha\rangle$.

In the case of $|\alpha, m\rangle$ the annihilation operator given by equation (57) has two vacua, namely, the vacuum and the m-photon state. When m is made negative in equation (57) the operator annihilates only the vacuum state $|0\rangle$. Setting $p = 1, j = 0$ and $\hat{B} = (1 + \frac{m}{1+\hat{a}^\dagger\hat{a}})\hat{a}$ in equation (135), we find that the corresponding raising operator is $\hat{G}^\dagger = \hat{a}^\dagger \frac{1+\hat{a}^\dagger\hat{a}}{1+m+\hat{a}^\dagger\hat{a}}$. Hence, the state $|\alpha, -m\rangle$ is written as

$$|\alpha, -m\rangle = e^{\alpha\hat{G}^\dagger}|0\rangle. \quad (68)$$

The state $|\alpha, -m\rangle$ is obtained by deforming the vacuum state $|0\rangle$ while the state $|\alpha, m\rangle$ is obtained from the m-photon state $|m\rangle$.

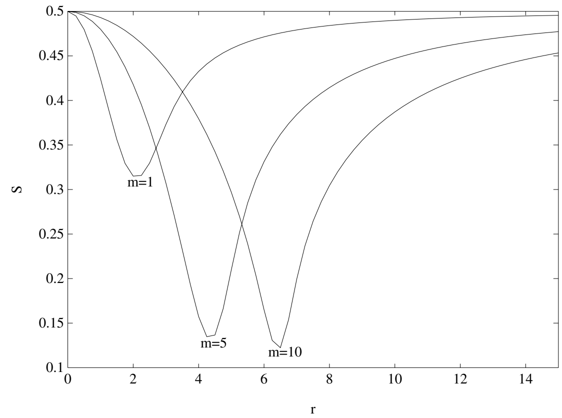

Figure 1. Uncertainty in p as a function of α(real) for $m = 1, 5$ and 10 for the state $|\alpha, -m\rangle$. The real α is represented as r.

2.3. Squeezing in $|\alpha, -m\rangle$

The state $|\alpha, -m\rangle$ exhibits squeezing in both x- and p-quadratures. The operators corresponding to the x- and p-quadratures are given in terms of $\hat{a}$ and $\hat{a}^\dagger$ by

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \qquad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}. \quad (69)$$

The expectation values of the relevant operators in the state $|\alpha, -m\rangle$ are

$$\langle\hat{a}\rangle = \alpha N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} \frac{(n+1)}{(n+m+1)}, \quad (70)$$

$$\langle\hat{a}^2\rangle = \alpha^2 N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} \frac{(n+1)(n+2)}{(n+m+1)(n+m+2)}, \quad (71)$$

and

$$\langle\hat{a}^\dagger\hat{a}\rangle = N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n. \quad (72)$$

The expectation values of $\hat{a}^\dagger$ and $\hat{a}^{\dagger 2}$ are obtained by taking the complex conjugates of $\langle\hat{a}\rangle$ and $\langle\hat{a}^2\rangle$ respectively. The expectation value of $(\triangle\hat{p})^2$ is

$$\langle p^2\rangle - \langle p\rangle^2 = \tfrac{1}{2}[1 + 2\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle]. \quad (73)$$

In figure 1 the quantity $\langle(\triangle\hat{p})^2\rangle$ is shown as a function of real α for various values of m. As expected the uncertainty in p is close to $\frac{1}{2}$, the uncertainty in p for the vacuum state, when α is close to zero. In the case of $|\alpha, m\rangle$ the variance is close to $m + \frac{1}{2}$ when α is close to zero. For real values of α the p-quadrature is always squeezed for the state $|\alpha, -m\rangle$. For large values of α the variance in p approaches that of $|\alpha\rangle$. The depth of squeezing increases with increasing m. Also, the value of $|\alpha|$ for which the maximum squeezing occurs increases with increasing m.

The expectation values of $\hat{a}^2$ and $\hat{a}^\dagger\hat{a}$ given by equations (71) and (72), respectively, do not change when α is replaced by $-\alpha$. This implies that the uncertainty in x is the same for states $|\alpha, -m\rangle$ and $|-\alpha, -m\rangle$. This result is true for the $\hat{p}$-quadrature too. Thus, whenever the state $|\alpha, -m\rangle$

exhibits squeezing in a quadrature the state $|-\alpha, -m\rangle$ also exhibits the same amount of squeezing in the quadrature. It may be noted that replacing α by $-\alpha$ amounts to rotating by π in the complex plane.

If α is substituted with $\mathrm{i}\alpha$, the expression for the variance in $\hat{p}$ for the state $|\alpha, -m\rangle$ becomes that in $\hat{x}$ for the state $|\mathrm{i}\alpha, -m\rangle$. Multiplying by i effects rotation by $\pi/2$ in the complex plane. Since there is squeezing in p for real values of α, the states $|\alpha, -m\rangle$ will exhibit squeezing in x for imaginary values of α.

2.4. Photon statistics of $|\alpha, -m\rangle$

The photon number distribution $p(n)$ for the state $|\alpha, -m\rangle$ is

$$p(n) = |\langle n|\alpha, -m\rangle|^2, \\ = N^2 m!^2 \frac{|\alpha|^{2n} n!}{(n+m)!^2}. \tag{74}$$

For $m = 0$ the distribution becomes a Poissonian distribution whose mean is $|\alpha|^2$.

The mean and the variance are numerically equal for a Poisson distribution. A measure of deviation from this behaviour is given by the Mandel q-parameter [49],

$$q = \frac{\langle(\triangle\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle}. \tag{75}$$

The value of q is zero for the CSs of the harmonic oscillator. A negative value of q indicates that the distribution $p(n)$ is sub-Poissonian. The PACSs $|\alpha, m\rangle$ exhibit sub-Poissonian statistics for all values of m. For the state $|\alpha, -m\rangle$ the mean values of $\hat{n}$ and $\hat{n}^2$ are given by

$$\langle\hat{n}\rangle = N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n, \tag{76}$$

$$\langle\hat{n}^2\rangle = N^2 m!^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n^2. \tag{77}$$

In figure 2 we have shown the q-parameter, calculated using equations (75)–(77), as a function of $|\alpha|$ for the state $|\alpha, -m\rangle$. The value of the q-parameter is always greater than zero, indicating the super-Poissonian nature of the photon distribution. For small values of α the q-parameter is close to zero. This is to be expected as $|\alpha, -m\rangle$ becomes the usual CS $|\alpha\rangle$ as $\alpha \to 0$. It is interesting to note that action of $\hat{a}^\dagger$ on any NCS yields another NCS [50].

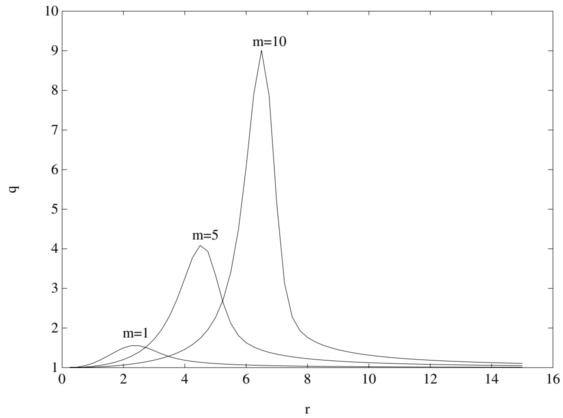

Figure 2. Mandel's q-parameter as a function of $|\alpha|$ for $m = 1, 5$ and 10. $|\alpha|$ is represented as r.

3. Even and odd nonlinear coherent states

In this section we extend the notion of even and odd CSs to the case of NCSs. From the definition of even and odd NCSs it is evident that they are eigenstates of the square of the annihilation operator. The operators $\hat{a}^2$, $\hat{a}^{\dagger 2}$ and $\hat{a}^\dagger\hat{a}$ form a closed algebra, i.e. the commutator of any two of these operators is the remaining operator but for a multiplicative constant. As a consequence a unitary operator $S(z)$ can be constructed as

$$\hat{S(z)} = \exp[\tfrac{1}{2}(z\hat{a}^{\dagger 2} - z^*\hat{a}^2)], \tag{78}$$

where z is complex. There are two vacua for $\hat{a}^2$, namely, the ground state $|0\rangle$ and the first excited state $|1\rangle$ and the corresponding CSs are obtained by the action of $\hat{S(z)}$ on the vacua. The number state expansions [51] of these CSs are

$$|z, 0\rangle = \hat{S(z)}|0\rangle \\ = (\cosh|z|)^{-1/2} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\mathrm{e}^{\mathrm{i}\theta}\tanh|z|)^n |2n\rangle, \tag{79}$$

and

$$|z, 1\rangle = \hat{S(z)}|1\rangle = (\sinh|z|)^{-1/2} \\ \times \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} (\mathrm{e}^{\mathrm{i}\theta}\tanh|z|)^n |2n+1\rangle. \tag{80}$$

Here $z \in \mathcal{C}$. These states are *not* the eigenstates of $\hat{a}^2$ as $[\hat{a}^2, \hat{a}^{\dagger 2}] \neq 1$. The state $|z, 0\rangle$ is a MUS. However, the uncertainties of the individual quadratures $\hat{x}$ and $\hat{p}$ are not the same as those of the vacuum state. Uncertainty in one of the quadratures is squeezed below the vacuum limit at the cost of increased uncertainty in the other quadrature and hence the state is nonclassical. The operator $\hat{S(z)}$ is called the *squeeze operator* and $|z, 0\rangle$ is the *squeezed vacuum* [52, 53]. This state has been extensively studied as it has reduced noise in one of the quadratures. Because of this special property, it has been shown to be of use in gravitational wave detection [54], enhancement and suppression of spontaneous emission [55], optical communication [56–58] etc. The ECSs and OCSs can also be thought of as the CSs for the $SU(1, 1)$ algebra satisfied by the operators $\hat{a}^2$, $\hat{a}^{\dagger 2}$ and $\hat{a}^\dagger\hat{a}$.

A possible extension of the notion of the even and odd CSs to NCSs is to define states which are linear combinations of $|\alpha, f\rangle$ and $|-\alpha, f\rangle$ [59, 61]. While the states $|\alpha, f\rangle$ and $|-\alpha, f\rangle$ are eigenstates of $f(\hat{n})\hat{a}$, their linear combinations are eigenstates of $(f(\hat{n})\hat{a})^2$. Another way to generalize the concept of even and odd CSs is to consider the eigenstates of a deformed two-photon annihilation operator, the deformation being premultiplying $\hat{a}^2$ by an operator-valued function of the number operator [62]. Even and odd NCSs are defined as the eigenstates of the operator $F(\hat{n})\hat{a}^2$, where $F(\hat{n})$ is an operator-valued function of the number operator $\hat{n}$. We denote the eigenstates as $|\alpha, F\rangle$ and they satisfy

$$F(\hat{n})\hat{a}^2|\alpha, F\rangle = \alpha|\alpha, F\rangle, \qquad \alpha \in \mathcal{C}. \tag{81}$$

Expanding $|\alpha, F\rangle$ in terms of the Fock states $|n\rangle$ of the harmonic oscillator as

$$|\alpha, F\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{82}$$

and substituting in equation (81) yields

$$c_{2n} = \alpha^n \frac{c_0}{F(2(n-1))!!\sqrt{(2n)!}}, \tag{83}$$

$$c_{2n+1} = \alpha^n \frac{c_1}{F(2n-1)!!\sqrt{(2n+1)!}}, \tag{84}$$

where $F(2(n-1))!! = F(0)F(2)F(4)\ldots F(2(n-1))$ and $F(2n-1)!! = F(1)F(3)F(5)\ldots F(2n-1)$. Here c_0 and c_1 are constants to be fixed by normalization of the states $|\alpha, F\rangle$.

If we choose $c_1 = 0$ (respectively, $c_0 = 0$), the state $|\alpha, F\rangle$ involves the superposition of even (respectively, odd) number states and represents the even (respectively, odd) NCS (ENCS (ONCS)). The ENCS is denoted as $|\alpha, F, +\rangle$ and the ONCS as $|\alpha, F, -\rangle$.

The ENCS has the number state expansion

$$|\alpha, F, +\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{F(2(n-1))!!\sqrt{(2n)!}} |2n\rangle, \tag{85}$$

with

$$|c_0|^{-2} = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[F(2(n-1))!!]^2 (2n)!}. \tag{86}$$

The state is normalizable provided the constant $|c_0|$ is nonzero and finite. This means that the terms in the summation for $|c_0|^{-2}$ should be such that

$$|\alpha|^2 \leqslant \lim_{n\to\infty} [F(2n)]^2 (2n+1)(2n+2). \tag{87}$$

If $F(n)$ decreases faster than n^{-1} for large n, then the range of α for which the state $|\alpha, F, +\rangle$ is normalizable is restricted to values satisfying equation (87) and in other cases the range of α is unrestricted.

The number state expansion for the ONCS is

$$|\alpha, F, -\rangle = c_1 \sum_{n=0}^{\infty} \frac{\alpha^n}{F(2n-1)!!\sqrt{(2n+1)!}} |2n+1\rangle. \tag{88}$$

Normalization of the state gives

$$|c_1|^{-2} = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[(F(2n-1))!!]^2 (2n+1)!}. \tag{89}$$

The range of α for which the ONCS is defined is given by the inequality

$$|\alpha|^2 \leqslant \lim_{n\to\infty} [F(2n+1)]^2 (2n+2)(2n+3). \tag{90}$$

In the linear limit, $F(\hat{n}) = 1$, the ENCS (respectively, ONCS) becomes the ECS (respectively, OCS) and the range of α is unrestricted. Depending on the form of $F(\hat{n})$ the even and odd NCS states may exhibit many of the nonclassical features. Now, we choose specific forms for $F(\hat{n})$ and discuss the properties of their respective even and odd NCS.

3.1. Even and odd nonlinear coherent states with $F(\hat{n}) = \frac{1}{1+k\hat{n}}$

If we take the operator function $F(\hat{n})$ to be $\frac{1}{1+k\hat{n}}$, where $k \geqslant 0$, the number state expansion for the ENCS, using equation (85), is

$$\begin{aligned} |\alpha, F, +1\rangle &= c_{+1} \sum_{n=0}^{\infty} \frac{\alpha^n (1+k(2n-2))!!}{\sqrt{(2n)!}} |2n\rangle, \\ |c_{+1}|^{-2} &= \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{((1+k(2n-2))!!)^2}{(2n)!}. \end{aligned} \tag{91}$$

The ONCS has the number state expansion

$$|\alpha, F, -1\rangle = c_{-1} \sum_{n=0}^{\infty} \frac{\alpha^n (1+k(2n-1))!!}{\sqrt{(2n+1)!}} |2n+1\rangle, \tag{92}$$

$$|c_{-1}|^{-2} = \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{((1+k(2n-1))!!)^2}{(2n+1)!}. \tag{93}$$

Using $F(n) = 1/(1+k\hat{n})$ in equations (87) and (90), the range of α is obtained as $|\alpha| \leqslant 1/k$.

In the linear limit of $F(\hat{n})$, obtained by setting $k = 0$, the state $|\alpha, F, +1\rangle$ becomes the ECS and the state is defined for all values of α. In the limit of k becoming unity the state $|\alpha, F, +1\rangle$ becomes

$$|\alpha, F, +1\rangle = c_{+1} \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(2n)!}}{2^n n!} |2n\rangle, \tag{94}$$

and the state is defined for $|\alpha| < 1$. It is interesting to note that the above state is the squeezed vacuum $|z, 0\rangle$ defined in equation (79). Comparing equations (79) and (94) we find that

$$|\alpha, F, +1\rangle = |z, 0\rangle, \tag{95}$$

where the eigenvalue α is related to z by

$$\alpha = e^{i\theta} \tanh|z|. \tag{96}$$

Here θ is the argument of z. The transformation given by equation (96) maps the unit disc on the α-plane to the whole of the z-plane. This implies that the range of the squeeze parameter z is not restricted as the entire z-plane is scanned by the transformation.

3.2. Squeezing properties of $|\alpha, F, +1\rangle$

The state $|\alpha, F, +1\rangle$ has the ECS and the squeezed vacuum as its limiting cases. It is natural, therefore, to study the squeezing property of the states for cases other than the special limits indicated above. For the states $|\alpha, F, +1\rangle$ the expectation values of $\hat{a}$ and $\hat{a}^\dagger$ vanish and we have

$$\langle \hat{a}^\dagger \hat{a}\rangle = |c_{+1}|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}[(1+k(2n-2))!!]^2}{(2n)!} 2n, \tag{97}$$

$$\langle \hat{a}^2\rangle = \alpha |c_{+1}|^2 \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{(1+k(2n-2))!!(1+2kn)!!}{(2n)!}. \tag{98}$$

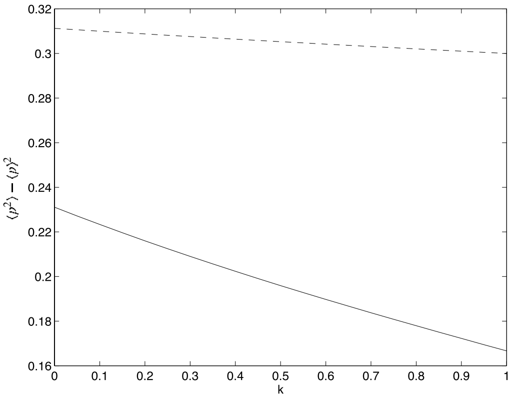

Figure 3. $\langle p^2\rangle - \langle p\rangle^2$ as a function of k for $\alpha = 0.5$ for the state $|\alpha, F, +1\rangle$.

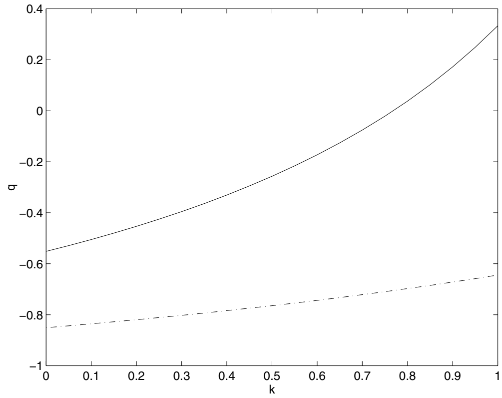

Figure 4. Mandel's q-parameter as a function of k for $\alpha = 0.5$ for the state $|\alpha, F, -2\rangle$.

The expectation value of $\langle \hat{a}^{\dagger 2}\rangle$ is the complex conjugate of $\langle \hat{a}^2\rangle$.

In figure 3 the variation of $\langle(\triangle\hat{p})^2\rangle$ with respect to k is shown when $\alpha = 1/2$. It is clear from figure 3 that the states $|\alpha, F, +1\rangle$ exhibit squeezing and the depth of squeezing increases with increasing k. As k varies from zero to unity the amount of squeezing in $\hat{p}$ for the state $|\alpha, F, +1\rangle$ varies from that of the ECS to that of the squeezed vacuum. The squeezing properties of the states $|\alpha, F, +1\rangle$ are intermediate between the ECS and the squeezed vacuum. The squeezed vacuum limit is reached when $k = 1$. Increasing k beyond unity squeezes $\langle(\triangle\hat{p})^2\rangle$ further.

3.3. Even and odd nonlinear coherent states with $F(\hat{n}) = \frac{1}{2+k\hat{n}}$

If we set $F(\hat{n})$ to be $\frac{1}{2+k\hat{n}}$ with $k \geqslant 0$, the eigenstates satisfy

$$\frac{1}{2+k\hat{n}}\hat{a}^2|\alpha, F, \pm 2\rangle = \alpha|\alpha, F, \pm 2\rangle, \qquad (99)$$

where + and − indicate the even and odd NCSs respectively.

The ENCS in terms of the number states $|n\rangle$ is

$$\begin{aligned} |\alpha, F, +2\rangle &= c_{+2}\sum_{n=0}^{\infty}\frac{\alpha^n(2+k(2n-2))!!}{\sqrt{(2n)!}}|2n\rangle, \\ |c_{+2}|^{-2} &= \sum_{n=0}^{\infty}|\alpha|^{2n}\frac{((2+k(2n-2))!!)^2}{(2n)!}. \end{aligned} \qquad (100)$$

and the ONCS is

$$|\alpha, F, -2\rangle = c_{-2}\sum_{n=0}^{\infty}\frac{\alpha^n(2+k(2n-1))!!}{\sqrt{(2n+1)!}}|2n+1\rangle, \qquad (101)$$

$$|c_{-2}|^{-2} = \sum_{n=0}^{\infty}|\alpha|^{2n}\frac{((2+k(2n-1))!!)^2}{(2n+1)!}. \qquad (102)$$

The even and odd NCSs defined above are normalizable for those values of α which satisfy $|\alpha| \leqslant 1/k$.

In the linear limit of $F(\hat{n})$, i.e. when $k = 0$, the state $|\alpha, F, -2\rangle$ becomes the OCS. In the limit of k becoming unity the state $|\alpha, F, -2\rangle$ becomes

$$|\alpha, F, -2\rangle = c_{-2}\sum_{n=0}^{\infty}\frac{\alpha^n\sqrt{(2n+1)!}}{2^n n!}|2n+1\rangle. \qquad (103)$$

Transforming from α to z using equation (96) it is evident that $|\alpha, F, -2\rangle$ is the squeezed first excited state defined in equation (80).

3.4. Statistical properties of $|\alpha, F, -2\rangle$

The photon number distribution of the OCS is oscillatory, becoming zero for even values of n. The photon number distribution $p(n)$ for the ONCS $|\alpha, F, -2\rangle$ is

$$p(2n+1) = |c_{-2}|^2\frac{|\alpha|^{2n}[(2+k(2n-1))!!]^2}{(2n+1)!}, \qquad (104)$$

$$p(2n) = 0. \qquad (105)$$

When $k = 1$ the distribution is that of the squeezed first excited state and when $k = 0$ the distribution is that of the OCS.

In figure 4 the q-parameter of the state $|\alpha, F, -2\rangle$ is shown as a function of k for $\alpha = 1/2$. It is seen that for a range of k the q parameter becomes negative, implying that the states given by equation (101) exhibit sub-Poissonian statistics in their photon number distribution for some values of k. With respect to the q-parameter, the states $|\alpha, F, -2\rangle$ are intermediate between the OCS, whose photon number distribution is sub-Poissonian, and the squeezed first excited state, whose photon number distribution is not sub-Poissonian.

3.5. Relation to other two-photon coherent states

As we have seen, the squeezed vacuum and the squeezed first excited states can be classified as even and odd NCSs respectively. New classes of two-photon CSs have been defined [48] by constructing the eigenstates of $\hat{a}^{\dagger -1}\hat{a}$ and $\hat{a}\hat{a}^{\dagger -1}$. Considering the action of the inverse operators on number states, as given in equations (64) and (65), we recognize the following:

$$\hat{a}^{-1} = \hat{a}^{\dagger}\frac{1}{1+\hat{a}^{\dagger}\hat{a}} \qquad (106)$$

$$\hat{a}^{\dagger -1} = \frac{1}{1+\hat{a}^{\dagger}\hat{a}}\hat{a}. \qquad (107)$$

Therefore,

$$\hat{a}^{\dagger -1}\hat{a} = \frac{1}{1+\hat{a}^\dagger\hat{a}}\hat{a}^2, \quad (108)$$

$$\hat{a}\hat{a}^{\dagger -1} = \frac{1}{2+\hat{a}^\dagger\hat{a}}\hat{a}^2. \quad (109)$$

These relations, in turn, imply that the eigenstates of $\hat{a}^{\dagger -1}\hat{a}$ (respectively, $\hat{a}\hat{a}^{\dagger -1}$) are the special cases of $|\alpha, F, \pm 1\rangle$ (respectively, $|\alpha, F, \pm 2\rangle$) when $k = 1$.

4. Generation of even and odd nonlinear coherent states

A simple and solvable model to describe the atom–field interaction was proposed by Jaynes and Cummings [30], in which the atom is treated as a dipole of constant dipole-moment. This model assumes that the spatial variation of the field is insignificant over the dimensions of the dipole, and hence for practical purposes the field can be taken to be constant. An appropriate model for the interaction should include the influence of the spatially varying electric field on the dipole moment of the atom [63]. A system where the spatial variation must be included is the case of a two-level ion trapped in an external harmonic oscillator potential. Apart from interacting with the external potential the two-level ion interacts with external laser fields also. The spatial variation of the electric field due to the lasers can be tailored so as to have different forms for the interaction Hamiltonian. This, in turn, makes the dipole moment of the ion a function of position. Inclusion of such a position dependent dipole moment gives rise to ion–field interactions which depend nonlinearly on the number operator $\hat{n}$ for the vibronic state of the ion in the external harmonic trap. This system has been studied in very many contexts: NCS [43], vibronic Jaynes–Cummings interaction [64], nonlinear Jaynes–Cummings interaction [65], generation of even and odd CS [66], quantum signatures of chaos [67], quantum nondemolition measurements [68], quantum logic operations [70], engineering of Hamiltonian [69] and generation of amplitude-squared squeezed states [71]. In this section the possibility of generating a class of even and odd NCSs in the above system is explored. It turns out that the stationary states of the vibronic motion (in the harmonic trap) of the ion are indeed even or odd NCSs [72]. The nonclassical properties of the ENCS and ONCS produced by the system are studied.

4.1. Description of the system

Consider a two-level ion, having an electronic transition frequency ω and a lower (second) vibrational sideband with respect to that frequency, trapped in a harmonic oscillator potential of frequency ν. Two laser fields, one of frequency ω and the other corresponding to the vibrational sideband transition frequency $\omega - 2\nu$, interact with the ion. The Hamiltonian of this system is

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}(t), \quad (110)$$

with

$$\hat{H}_0 = \hbar\nu\hat{a}^\dagger\hat{a} + \hbar\omega\hat{\sigma}_{22}. \quad (111)$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are respectively the raising and lowering operators for the eigenstates of the external harmonic trap. The internal states, namely, the ground state $|{\rm g}\rangle$ and the excited state $|{\rm e}\rangle$, are the eigenstates of $\hat{\sigma}_{22}$ with respective eigenvalues zero and ω. The free Hamiltonian $\hat{H}_0$ describes the free motion of the internal and external degrees of freedom. Here internal degree of freedom refers to the electronic levels of the two-level ion and the external degree of freedom refers to the vibrational state of the ion in the external harmonic oscillator potential. The interaction Hamiltonian

$$\hat{H}_{\rm int}(t) = \lambda[E_0{\rm e}^{[-{\rm i}(k_0\hat{x}-\omega t)]} + E_1{\rm e}^{[-{\rm i}(k_1\hat{x}-(\omega-2\nu)t)]}]\hat{\sigma}_- + {\rm h.c.}, \quad (112)$$

describes the interaction of the ion with the two laser fields. The operator $\hat{\sigma}_-$ and its adjoint $\hat{\sigma}_+$ are the electronic flip operators corresponding to the transition $|{\rm e}\rangle \rightarrow |{\rm g}\rangle$ and $|{\rm g}\rangle \rightarrow |{\rm e}\rangle$ respectively. These operators are defined by their action on the internal states of the ion:

$$\hat{\sigma}_-|{\rm g}\rangle = 0, \qquad \hat{\sigma}_-|{\rm e}\rangle, = |{\rm g}\rangle \quad (113)$$

$$\hat{\sigma}_+|{\rm e}\rangle = 0, \qquad \hat{\sigma}_+|{\rm g}\rangle = |{\rm e}\rangle. \quad (114)$$

The constant λ is the electronic coupling matrix element and k_0, k_1 are the wavevectors of the laser fields. The operator $\hat{x}$ for the position of the centre of mass of the ion is

$$\hat{x} = \frac{\eta}{k_L}(\hat{a} + \hat{a}^\dagger), \quad (115)$$

where η is the Lamb–Dicke (LD) parameter and $k_L \simeq k_0 \simeq k_1$ is the wavevector of the driving laser field. The LD parameter is defined as

$$\eta = k_L\sqrt{\hbar/(2M\nu)}, \quad (116)$$

where M is the mass of the ion. The frequency of the laser field of amplitude E_0 is the same as the electronic transition frequency ω. The second laser field of amplitude E_1 is of frequency $\omega - 2\nu$ and this corresponds to the second vibrational sideband frequency. Note that the operator $\hat{x}$ that occurs in the exponentials multiplying the field amplitudes is the centre-of-mass position operator. It is important to realize that the external laser fields are treated classically. With these definitions the interaction Hamiltonian becomes

$$\begin{aligned}\hat{H}_{\rm int}(t) = \lambda[&E_0{\rm e}^{[-{\rm i}(\eta(\hat{a}^\dagger+\hat{a})-\omega t)]}\\ &+E_1{\rm e}^{[-{\rm i}(\eta(\hat{a}^\dagger+\hat{a})-(\omega-2\nu)t)]}]\hat{\sigma}_- + {\rm h.c.}\end{aligned} \quad (117)$$

Expanding the exponentials in the Hamiltonian as a power series in η and retaining only first-order terms in η gives the usual Jaynes–Cummings Hamiltonian for the ion–field interaction.

The fast and slow rotating terms in the Hamiltonian can be identified by writing it in a suitable interaction picture. The interaction picture is defined by the unitary transformation $\exp(-\frac{{\rm i}\hat{H}_0 t}{\hbar})$ and under this transformation the interaction Hamiltonian $\hat{H}_{\rm int}$ becomes

$$\hat{H}'_{\rm int} = \exp\left(-\frac{{\rm i}\hat{H}_0 t}{\hbar}\right)\hat{H}_{\rm int}(t)\exp\left(\frac{{\rm i}\hat{H}_0 t}{\hbar}\right), \quad (118)$$

$$= \hbar\Omega_1\exp(-\eta^2/2)\hat{\sigma}_-\hat{G} + {\rm h.c.} \quad (119)$$

in which

$$\hat{G} = \left[\sum_{k,l=0}^{\infty} \frac{(\mathrm{i}\eta)^{k+l}}{k!l!} \mathrm{e}^{\mathrm{i}(k-l-2)\nu t} \hat{a}^{\dagger k} \hat{a}^{l} + \frac{\Omega_0}{\Omega_1} \times \sum_{k,l=0}^{\infty} \frac{(\mathrm{i}\eta)^{k+l}}{k!l!} \mathrm{e}^{\mathrm{i}(k-l)\nu t} \hat{a}^{\dagger k} \hat{a}^{l} \right]. \tag{120}$$

Here $\Omega_i = \frac{\lambda E_i}{\hbar}$ $(i = 1, 2)$ are the Rabi frequencies of the two laser fields tuned to the electronic transition and the second sideband respectively.

The interaction Hamiltonian of equation (119) is exact. Now, we make the RWA with respect to the vibronic frequency ν. This amounts to neglecting the terms rotating with frequencies ν or more. Under this approximation, the interaction picture Hamiltonian becomes

$$\hat{H}'_{\mathrm{int}} = \hbar\Omega_1 \exp(-\eta^2/2)\hat{\sigma}_+ \hat{F} + \mathrm{h.c.}, \tag{121}$$

with

$$\hat{F} = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\eta)^{2k+2}}{k!(k+2)!} \hat{a}^{\dagger k} \hat{a}^{k+2} + \frac{\Omega_0}{\Omega_1} \sum_{k=0}^{\infty} \frac{(\mathrm{i}\eta)^{2k}}{k!^2} \hat{a}^{\dagger k} \hat{a}^{k}. \tag{122}$$

The Hamiltonian given in equation (121) describes the ion–field interaction after making the vibronic RWA.

4.2. Time evolution of the system

The time evolution of the system (the two-level ion interacting with the two external laser fields), when coupled to an external harmonic oscillator bath at absolute zero, is governed by the master equation for its density operator $\hat{\rho}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{\mathrm{i}}{\hbar}[\hat{H}'_{\mathrm{int}}, \hat{\rho}] + \frac{\Gamma}{2}(2\hat{\sigma}_- \hat{\rho}' \hat{\sigma}_+ - \hat{\sigma}_{22}\hat{\rho} - \hat{\rho}\hat{\sigma}_{22}). \tag{123}$$

The external laser fields in the interaction Hamiltonian are treated as classical fields and hence, in the limit of vanishing amplitudes for the external fields, there will be no spontaneous emission. The vacuum fluctuations are required to induce the atom to emit spontaneously. The second term in the master equation accounts for the spontaneous decay in a phenomenological way. The energy relaxation rate via spontaneous emission is Γ and

$$\hat{\rho}' = \tfrac{1}{2} \int_{-1}^{1} \mathrm{d}y\, W(y) \mathrm{e}^{\mathrm{i}\eta(\hat{a}+\hat{a}^\dagger)y} \hat{\rho} \mathrm{e}^{-\mathrm{i}\eta(\hat{a}+\hat{a}^\dagger)y}, \tag{124}$$

accounts for changes in the vibrational energy due to spontaneous emission. $W(y)$ gives the angular distribution of spontaneous emission.

The steady state solution $\hat{\rho}_s$ of equation (123) is obtained by setting $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = 0$ and it satisfies

$$\frac{\mathrm{i}}{\hbar}[\hat{H}'_{\mathrm{int}}, \hat{\rho}_s] = \frac{\Gamma}{2}(2\hat{\sigma}_- \hat{\rho}'_s \hat{\sigma}_+ - \hat{\sigma}_{22}\hat{\rho}_s - \hat{\rho}_s\hat{\sigma}_{22}). \tag{125}$$

ρ'_s is obtained from equation (124) by replacing ρ with ρ_s. To solve equation (125) we make the ansatz that $\hat{\rho}_s$ is given by

$$\hat{\rho}_s = |\mathrm{g}\rangle|\zeta\rangle\langle\zeta|\langle g|, \tag{126}$$

where $|\zeta\rangle$ is the vibrational state of the ion. Using equations (121) and (126) in the master equation (125), we arrive at the following condition on the state $|\zeta\rangle$:

$$\hat{F}|\zeta\rangle = 0. \tag{127}$$

Making use of the operator expansion for $\hat{F}$ given by equation (122) we obtain

$$\langle n+2|\zeta\rangle = \frac{\Omega_0}{\Omega_1\eta^2} \frac{(n+1)(n+2)L_n^0(\eta^2)}{\sqrt{(n+1)(n+2)}L_n^2(\eta^2)} \langle n|\zeta\rangle, \tag{128}$$

where L_n^m is an associated Laguerre polynomial defined by

$$L_n^m(x) = \sum_{l=0}^{n} \binom{n+m}{n-l} \frac{(-x)^l}{l!}. \tag{129}$$

Here $\langle n+2|\zeta\rangle$ are the expansion coefficients for the state $|\zeta\rangle$ in the Fock states basis. Comparing with equations (83), (84) indicates that the state $|\zeta\rangle$ is an even or odd NCS with

$$\alpha = \frac{\Omega_0}{\Omega_1\eta^2}, \tag{130}$$

and

$$F(n) = L_n^2(\eta^2)[(n+1)(n+2)L_n^0(\eta^2)]^{-1}. \tag{131}$$

In the limit $\eta \to 0$ the function $F(n) \to 1/2$ for all n. Hence in the small-η limit the even and odd NCSs become the ECS and OCS respectively. In particular, the state $|\alpha, F, +\rangle$ (respectively, $|\alpha, F, -\rangle$) becomes $|2\alpha, +\rangle$ (respectively, $|2\alpha, -\rangle$) in the limit $\eta \to 0$.

The master equation, which governs the time evolution of the system, only contains the even powers of $\hat{a}$ and $\hat{a}^\dagger$. This ensures that the parity of the initial vibronic state of the system is preserved during the time evolution. If the initial vibronic state of the ion is a combination of even (respectively, odd) number states then the state of the system at later times will only involve a superposition of even (respectively, odd) number states.

4.3. The even nonlinear coherent state and its properties

If the initial state of the ion is the vacuum state then the stationary state of the system is an ENCS given by

$$|\alpha, F, +\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(2n)!}F(2n-2)!!} |2n\rangle, \qquad N^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(2n)!(F(2n-2)!!)^2}}, \tag{132}$$

where α and $F(n)$ are defined by equations (130) and (131) respectively. This state is the ENCS for the vibrational motion of the centre of mass of the ion in the harmonic potential. The behaviour of the expansion coefficients $\langle n|\alpha, F, +\rangle$ is highly oscillatory, becoming zero for odd n. This oscillatory behaviour is one of the nonclassical features.

In figure 5 the uncertainty in p is shown as a function of η. From the figure it is clear that the states exhibit squeezing in p quadrature.

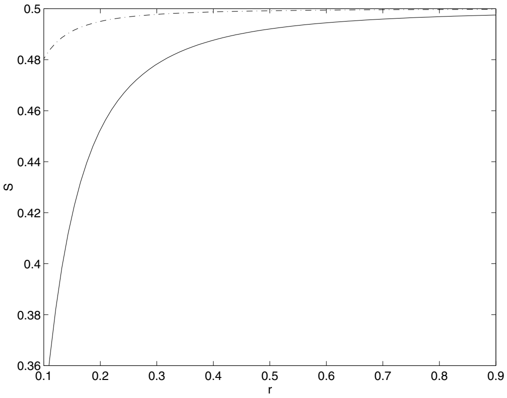

Figure 5. $\langle p^2\rangle - \langle p\rangle^2$ as a function of η for $\frac{\Omega_0}{\Omega_1} = 0.001$ (solid curve) and 0.0001 (dashed curve) for the state $|\alpha, F, +\rangle$. η is represented as r.

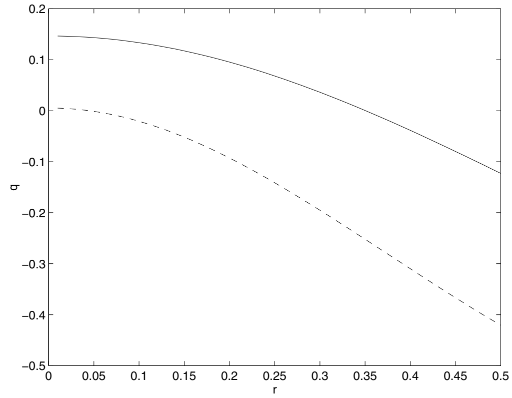

Figure 7. Mandel's q-parameter as a function of η for the state $|\alpha, F, +\rangle$. The solid curve is for $\alpha = 1.0$ and the dashed curve for $\alpha = 2.0$. r represents η.

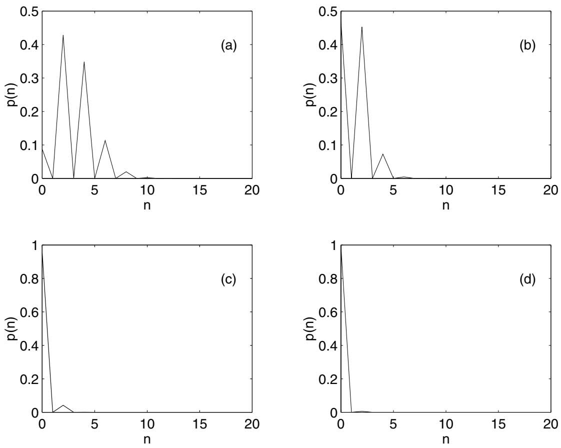

Figure 6. Occupation number distribution $p(n)$ as a function of n for the state $|\alpha, F, +\rangle$ for various η values and $\frac{\Omega_0}{\Omega_1} = 0.0001$: (a) $\eta = 0.008$, (b) $\eta = 0.012$, (c) $\eta = 0.02$ and (d) $\eta = 0.1$.

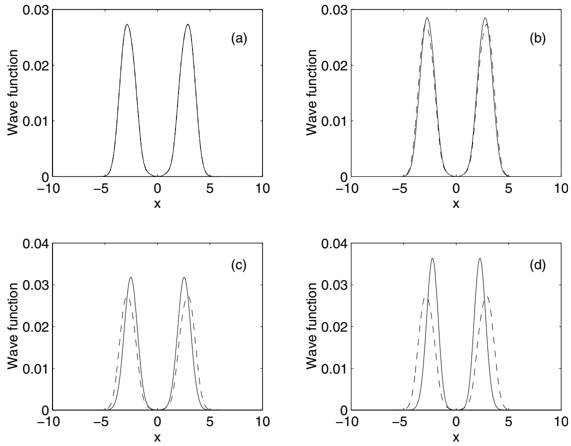

Figure 8. Position basis wavefunction corresponding to the state $|\alpha, F, +\rangle$ with $\alpha = 2.0$: (a) $\eta = 0.0$, (b) $\eta = 0.15$, (c) $\eta = 0.30$ and (d) $\eta = 0.5$. The dashed curve corresponds to the ECS wavefunction and is the same as (a).

As η increases the uncertainty in $\hat{p}$ approaches that of the vacuum state. The reason for this behaviour is the following. As η increases the occupation number distribution $p(n) = |\langle n|\alpha, F, +\rangle|^2$ starts peaking near $n = 0$. To make this explicit we have shown in figure 6 the occupation number distribution $p(n)$ as a function of n for various values of η.

The occupation number distribution for the ECS is always super-Poissonian, meaning that the variance in $\hat{n}$ is larger than its mean. For the ENCS defined by equation (132), the distribution $p(n)$ can have negative q for suitable values of η and α. In figure 7 we have shown the variation of q with respect to the LD parameter η for $\alpha = 1.0$. It is evident that the ENCS can have features which are absent from the ECS.

In the limit $\eta \to 0$ the ENCS becomes the ECS, which is a cat state. In actual experiments such a limit may be difficult to achieve. Therefore it is required to study the behaviour of the ENCS wavefunction for values other than $\eta = 0$. In figure 8 we have plotted the wavefunction of the ENCS, given by equation (132), as a function of the centre-of-mass coordinate x for various values of η. For the sake of comparison the ECS wavefunction (dashed curve) is also shown. It is seen that the ENCS of equation (132) is indeed close to the ECS. The deviation from the ECS wavefunction increases slowly as the value of η increases.

4.4. The odd nonlinear coherent state and its properties

If the initial state of the ion is the first excited state of the harmonic trap then the state of the system at later times will only involve odd number states. The resultant stationary state of the system is an ONCS given by

$$|\alpha, F, -\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(2n+1)!}F(2n-1)!!}|2n+1\rangle, \quad (133)$$

$$N^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(2n+1)!(F(2n-1)!!)^2}}, \quad (134)$$

where $F(n)$ and α are again defined by equations (130) and (131) respectively.

The occupation number distribution of the ONCS defined in equation (133) is oscillatory, becoming zero for even n, and exhibits sub-Poissonian character. Figure 9 shows the Mandel q-parameter as a function of η for the states given by equation (133). The state $|\alpha, F, -\rangle$, equation (133), exhibits sub-Poissonian statistics for those values of η for which q is negative. It is interesting to note that the q-parameter approaches -1 (the value of q for the first excited

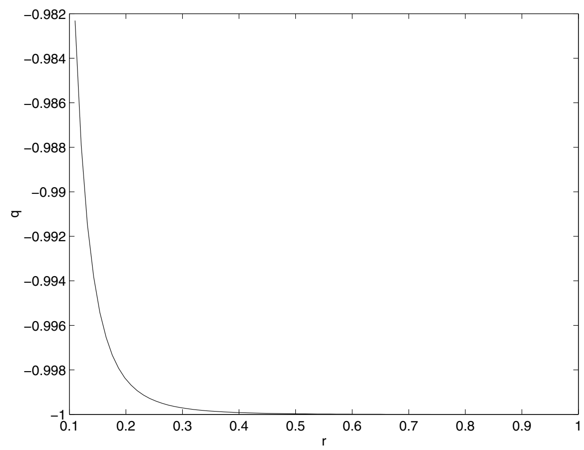

Figure 9. Mandel's q-parameter as a function of η for $\frac{\Omega_0}{\Omega_1} = 0.001$ for the state $|\alpha, F, -\rangle$. Here r represents η.

of the harmonic oscillator) when η becomes large, the reason being that the occupation number distribution peaks at $n = 1$ as η increases.

5. Summary

The concept of NCSs is a generalization of the algebraic definition of the CSs, namely, the eigenstates of the annihilation operator. This is a very apt definition of CSs for nonlinear systems, in particular for systems whose frequency depends on energy. An important and realizable example for NCSs in the context of electromagnetic fields is the PACSs. As an extension of the concept of even and odd CSs, the even and odd NCSs are defined as the eigenstates of deformed two-photon annihilation operator. The squeezed vacuum and the squeezed first excited states can be interpreted as even and odd NCSs respectively with suitable choice for the deforming function. It is possible to realize such states in the vibrational motion of a harmonically trapped two-level ion interacting with external laser fields.

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Appendix

In this appendix a brief review of the method of Shantha *et al* [73] to construct raising operators for a given generalized annihilation operator is presented.

Consider an 'annihilation operator' $\hat{B}$ which annihilates a set of number states $|n_i\rangle, i = 1, 2, \ldots, k$. Then we can construct a sector S_i by repeatedly applying $\hat{B}^\dagger$, the adjoint of $\hat{B}$, on the number state $|n_i\rangle$. Thus we have k sectors corresponding to the states that are annihilated by $\hat{B}$. A given sector may turn out to be either finite or infinite dimensional. If a sector, say S_j, is of infinite dimension then we construct an operator $\hat{G_j}^\dagger$ such that the commutator $[\hat{B}, \hat{G_j}^\dagger] = 1$ holds in that sector. Then the eigenstates of $\hat{B}$ can be written as $e^{\alpha \hat{G_j}^\dagger}|n_j\rangle$. Let operator $\hat{B}$ be of the form $f(\hat{n})\hat{a}^p$, where p is a non-negative integer and $f(\hat{n})$ is an operator-valued function of the number operator $\hat{a}^\dagger\hat{a}$, such that it annihilates the number state $|j\rangle$. Then $\hat{G_j}^\dagger$ is constructed as

$$\hat{G_j}^\dagger = \frac{1}{p}\hat{B}^\dagger \frac{1}{\hat{B}\hat{B}^\dagger}(\hat{a}^\dagger\hat{a} + p - j). \tag{135}$$

Since $[\hat{B}, \hat{G_j}^\dagger] = 1$ we obtain another relation by taking the adjoint of the commutation relation and arrive at $[\hat{G}_j, \hat{B}^\dagger] = 1$. Thus another pair of raising and lowering operators on the sector S_j is generated.

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